Resonator Losses and Resonance Properties

4.1 Losses

In our discussion of resonators so far, we have ignored losses. Losses are both necessary and unavoidable. For example, to couple energy into the cavity, or to extract energy from it, mirror reflectivity must be less than unity. In addition, modes also suffer losses due to diffraction, scattering and absorption. These losses are specified for one round trip of the circulating field inside the cavity.

A. Transmission losses

These losses are incurred by the field due to finite transmittivity of the mirrors or absorption in their reflective coating. These losses are localized at the mirrors. When a wave circulating inside the cavity encounters a mirror of reflectivity $\mathcal{R}$, a fraction $\mathcal{R}$ of its power is reflected back into the cavity and a fraction $1 - \mathcal{R} = \mathcal{T} + \mathcal{A}$ is lost either due to mirror transmission (a fraction $\mathcal{T}$) or absorption (a fraction A) by the mirror coating. Mirror loss is usually dominated by the transmission loss. For this reason mirror losses are simply referred to as transmission losses.

In laser physics it is customary to characterize mirror transmission and absorption losses in terms of the loss factor, a dimensionless quantity, defined by

$$\mathcal{L} \equiv -\ln \mathcal{R} = -\ln(1 - \mathcal{T} - A)\,. \tag{4.1}$$

For small mirror transmission and absorption, the loss factor reduces to the fractional power loss suffered by the wave at the mirror

$$\mathcal{L} = -\ln(1 - \mathcal{T} - A) \approx \mathcal{T} + A\,. \tag{4.2}$$

For large mirror losses, $\mathcal{L}$ can exceed unity. For example for a mirror of 30% reflectivity, the loss factor is 1.20, whereas for a mirror of 95% reflectivity it is 0.05. In the latter case it is equal to the mirror transmittance $\mathcal{T} = 1 - \mathcal{R}$.

In a two mirror cavity the wave is reflected once from each of the mirrors so that in one round trip it has a fraction $\mathcal{R}_1\mathcal{R}_2$ of its initial power remaining inside the cavity and a fraction $1 - \mathcal{R}_1\mathcal{R}_2$ lost due to mirror transmission and absorption. The loss factor in this case is $\mathcal{L} = -\ln \mathcal{R}_1\mathcal{R}_2$. Generalizing this to an N-mirror cavity, we find the loss factor due to mirror transmission and absorption is

$$\mathcal{L}_{\scriptscriptstyle T} = -\ln[\mathcal{R}_1\mathcal{R}_2 \cdots \mathcal{R}_N] = \sum_{j=1}^{N} -\ln \mathcal{R}_j = \sum_{j=1}^{N} \mathcal{L}_j \tag{4.3}$$

B. Diffraction losses

An important parameter for discussing diffraction losses in finite diameter optical resonators is the resonator Fresnel number N_F, defined by

$$N_{\scriptscriptstyle F} = \frac{na^2}{L\lambda} \tag{4.4}$$

where $2a$ is the transverse width of the resonator end mirrors, L is the length of the resonator, λ is the wavelength of light and n is the refractive index of the medium filling the resonator. This number is the number of Fresnel zones across one end mirror as seen from the center of the opposite mirror. Let us recall that the end mirror spot size in a symmetric confocal resonator of length L is $w_1^2 = L\lambda/\pi n$ and other stable resonators have similar spot sizes at the mirrors. Then the expression for N_F can be written as

$$N_F = \frac{\pi n}{\pi L\lambda}\frac{a\pi}{\pi} = \frac{\pi a^2}{\pi w_1^2}\frac{1}{\pi}\,. \tag{4.5}$$

Thus, N_F is the ratio of the resonator mirror area to the area of the lowest order confocal mode area divided by π. If we note further, that the radius of an nth order Hermite-Gaussian mode is to a good approximation $\rho_n \approx \sqrt{n}\, w_1$. Then the largest order of Hermite-gaussian or Laguerre-gaussian mode that will fit within the mirror aperture is

$$n_{\max} = \frac{a^2}{w_1^2} = \pi N_F\,. \tag{4.6}$$

The exact calculation of losses must be done using Huygens-Kirchoff diffraction theory. These calculations give both the losses and additional phase shifts, which are the exact versions of the Guoy phase shifts $\psi(z)$ given in the ideal Gaussian limit by the $\psi(z) = \tan^{-1}(z/z_0)$ formula. They, thus, determine the exact spacing of transverse modes in the finite-aperture resonators. In practice, for $n_F \geq 10$, the gaussian approximation for the mode shape is excellent for the lowest few modes.

A number of empirical formulas for the one-way power loss per pass $\mathcal{L}_{\text{diff}}$ in finite-aperture resonators have been developed by various researchers.

Confocal square mirrors

$$\mathcal{L}_{\text{diff}} = \begin{cases} 8\pi\sqrt{2N_F}\, e^{-4\pi N_F} & N_F \geq \frac{1}{2} \\ 1 - 16N_F^2 e^{-8\pi^2 N_F^2/9} & N_F \to 0 \end{cases} \tag{4.7}$$

Confocal circular mirrors

$$\mathcal{L}_{\text{diff}} = \begin{cases} 16\pi^2 N_F\, e^{-4\pi N_F} & N_F \geq 1 \\ 1 - (\pi N_F) & N_F \to 0 \end{cases} \tag{4.8}$$

Planar strip mirrors

$$\mathcal{L}_{\text{diff}} = 0.12\, N_{_F}^{-3/2} \qquad N_{_F} \geq 1 \tag{4.9}$$

Planar circular mirrors

$$\mathcal{L}_{\text{diff}} = 0.33\, N_{_F}^{-3/2} \qquad N_{_F} \geq 1 \tag{4.10}$$

C. Absorption and scattering losses

These losses due to scattering and absorption in the medium are distributed throughout the cavity. They are also referred to as distributed or internal losses. Such losses are specified in terms of an absorption coefficient α, with units $[\alpha]$=m^{-1} or more practically cm^{-1}. After one round-trip, a wave circulating inside the cavity will have a fraction $e^{-\alpha 2L}$ of its initial power remaining inside the cavity and a fraction $1 - e^{-\alpha 2L}$ lost due to absorption. Here $2L$ is the round trip length of the cavity. Sometimes the factor $e^{-\alpha 2L}$ is treated as an effective reflectivity and internal losses per pass are sepcified in terms of an effective transmittance $\mathcal{T}_i \equiv 1 - e^{-\alpha 2L}$. The loss factor for absorptive losses is then defined in a manner analogous to transmission losses

$$\mathcal{L}_i \equiv \begin{cases} -\ln \mathcal{R}_i = -\ln(1 - \mathcal{T}_i) & \text{losses specified as } \mathcal{T}_i \\ \ln[e^{-\alpha 2L}] = -\alpha 2L & \text{losses specified as } \alpha \end{cases} \tag{4.11}$$

The cavity loss factor is the sum of internal (distributed absorptive and scattering losses) and external loss (mirror transmission, absorption, and diffraction losses) factors

$$\mathcal{L} = \mathcal{L}_i + \mathcal{L}_e = -\ln \mathcal{R}_i - \ln \mathcal{R}_1 \mathcal{R}_2 \cdots \mathcal{R}_N \tag{4.12}$$

Cavity life Time

As a consequence of these losses any field inside the cavity will lose energy as it propagates inside the cavity. If there are no sources to replenish the lost energy, field amplitude and therefore the intensity will decay with a time constant determined by the losses.

To be concrete, let us consider a two mirror cavity of length L. Let $\mathcal{E}_0$ be the field amplitude and I_0 the corresponding intensity at time $t = 0$. Then one round trip distance is $2L$ and the time for one round trip is

$$\tau_{_R} = 2L/(c/n) = 2nL/c \tag{4.13}$$

After each roundtrip the intensity is reduced by the factor[1] $\mathcal{R}_1\mathcal{R}_2 e^{-2\alpha L}$. After m round trips the intensity will be

$$\begin{aligned} I(t_m = m\tau_{_R}) &= I_0 \left[\mathcal{R}_1\mathcal{R}_2 e^{-2\alpha L}\right]^m = I_0 \exp[-(2\alpha L - \ln \mathcal{R}_1\mathcal{R}_2)m] \\ &= I_0 \exp[-\frac{\mathcal{L}_c}{\tau_{_R}} \cdot m\tau_{_R}] \\ &= I_0 \exp[-t_m/\tau_c] \end{aligned} \tag{4.14}$$

where

$$t_m = m\tau_{_R} = m\frac{2nL}{c} \tag{4.15}$$

$$\tau_c = \frac{\tau_{_R}}{2\alpha L - \ln \mathcal{R}_1\mathcal{R}_2} = \frac{2nL}{c\mathcal{L}_c} \tag{4.16}$$

If we take the intensity (4.14) equation to hold for all times t, not just at discrete instants t_m, we can write it as

$$I(t) = I_0 \exp(-t/\tau_c)\,. \tag{4.17}$$

The light intensity in this approximation decays with a time constant τ_c given by Eq. (4.16). It is the time for the cavity intensity to decrease to $1/e \approx 37\%$ of its initial value. It is a measure of the time for which the

[1]Here we are ignoring diffraction losses which are usually small for large aperture mirrors. They are easily incorporated by including multiplicative factors such as $(1 - \mathcal{L}_{\text{diff}})$ with mirror reflectivities in Eq.(4.14)

cavity can store electromagnetic energy. Time τ_c is called the lifetime of the cavity. It is naturally related to the cavity loss factor $\mathcal{L}_c$.

For large mirror reflectivities $\mathcal{R}_1$, $\mathcal{R}_2 \approx 1$ and negligible absorption, we can write $\mathcal{R}_1 = 1-\mathcal{T}_1$ and $\mathcal{R}_2 = 1-\mathcal{T}_2$ where the mirror transmittances satisfy $\mathcal{T}_1\,,\mathcal{T}_2 \ll 1$. The expression for the cavity life time then simplifies to

$$\tau_c = \frac{2nL}{c(2\alpha L + \mathcal{T}_1 + \mathcal{T}_2)} \equiv \frac{2nL}{c\mathcal{L}} \tag{4.18}$$

where the total single pass loss $\mathcal{L} = 2\alpha L + \mathcal{T}_1 + \mathcal{T}_2$ is the sum of intrinsic (due to diffraction, absorption scattering etc.) loss per pass $2\alpha L$ and transmission losses

$$\mathcal{L} = \mathcal{L}_{_I} + \mathcal{L}_{_T}\,, \quad \mathcal{L}_{_I} = 2\alpha L\,, \quad \mathcal{L}_{_T} = \mathcal{T}_1 + \mathcal{T}_2\,. \tag{4.19}$$

These equations are easily generalized to cavities involving more than two mirrors.

Order of magnitude estimate Let us estimate τ_c for a cavity consisting of two mirrors of reflectivities $\mathcal{R}_1 = 0.98 = \mathcal{R}_2$ separated by 90 cm in air. Since the mirror reflectivities are large we can use Eq. (4.18) for τ_c to obtain

$$\tau_c = \frac{2nL}{c}\frac{1}{2\alpha L + \mathcal{T}_1 + \mathcal{T}_2} = \frac{2\times 0.90}{3.00\times 10^8}\times\frac{1}{0.04}\,\mathrm{s} = 150\,\mathrm{ns} \tag{4.20}$$

This differs by less than 2% from the result (τ_c = 148 ns) obtained by using Eq. (4.16).

The cavity lifetime, although short by ordinary standards, is typically long compared to an optical cycle. For $\lambda_o = 600$ nm an optical cycle is $\tau_0 = \lambda/c = 600\times 10^{-9}/3.00\times 10^8 = 2.00\times 10^{-15}$ s. The ratio $\tau_c\tau_0$ is then the number of cycles an optical field will execute in one cavity life time. For our example this ratio

$$\frac{\tau_c}{\tau_0} = \frac{150\times 10^{-9}}{2.00\times 10^{-15}} = 7.5\times 10^7 \tag{4.21}$$

is a very large number. Thus if a certain amount of energy is deposited in the cavity, it will take many millions of optical cycles for the cavity to empty out. The ratio τ_c/τ_0 is a measure of the quality of a resonator and is related to the Q-factor of the cavity

$$Q = 2\pi \times \frac{\tau_c}{\tau_0} = 2\pi \frac{2nL}{\lambda \mathcal{L}} \tag{4.22}$$

A cavity that stores energy for a long time (many optical cycles) is a high-Q (high quality) cavity. From the dependence of Q on cavity losses we see that lower the loss higher the quality. Expressing τ_c in terms of Q

$$\tau_c = Q\frac{\tau_0}{2\pi} = \frac{Q}{\omega_0} \tag{4.23}$$

we write the time dependence of the intensity as

$$I(t) = I_0 e^{-\omega_0 t/Q}\,. \tag{4.24}$$

Noting that the intensity is given by $I = (W/V)c/n$ where W is the stored and V is the effective mode volume, we can write an equation for the energy stored in the resonator

$$\frac{dW}{dt} = -\frac{\omega_0}{Q} W\,. \tag{4.25}$$

With the help of this equation we find that the energy lost in one optical cycle $\tau_0 = 2\pi/\omega_0$ is $\Delta W = \tau_0 \left|\frac{dW}{dt}\right| = \frac{2\pi W}{Q}$ so that Q can be written as

$$Q = 2\pi \frac{W}{\Delta W}\,. \tag{4.26}$$

Thus Q is 2π times the ratio of energy stored in the resonator to the energy lost in one optical cycle. Another way of stating this is that Q is 2π times the number of optical cycles it will take to lose the energy stored at a constant rate of ΔW per optical cycle. Note that Q can be quite high even for lossy cavities. For example, the Q-factor of 50 cm long cavity with $90\lambda = 600$ nm is

$$Q = 2\pi \frac{2nL}{\mathcal{L}\lambda} = 2\pi \frac{2 \times 0.50}{0.90 \times 600 \times 10^{-9}} = 1.1 \times 10^7 \tag{4.27}$$

Such large values of Q arise because although the loss per pass is large, the loss per optical cycle is still small.

Cavity linewidth (FWHM of the spectrum)

From the decay of the intensity ($I \propto |\mathcal{E}|^2$) it follows that the amplitude of cavity mode (frequency ω_0 will have the time dependence

$$\mathcal{E}(t) = \mathcal{E}_0 e^{-t/2\tau_c - i\omega_0 t}\,. \tag{4.28}$$

Taking the Fourier transform of the amplitude we find

$$\widetilde{\mathcal{E}}(\omega) = \int_0^\infty dt\, \mathcal{E}(t) e^{i\omega t} = \frac{\mathcal{E}_0}{i(\omega_0 - \omega) + 1/2\tau_c}\,. \tag{4.29}$$

This leads to the cavity mode power spectrum

$$\left|\widetilde{\mathcal{E}}(\omega)\right|^2 = \frac{\left|\widetilde{\mathcal{E}}_0\right|^2}{(\omega - \omega_0)^2 + (1/2\tau_c)^2} = \left|\widetilde{\mathcal{E}}_0\right|^2 \tau_c \left[\frac{1}{\pi} \frac{1/4\pi\tau_c}{(\nu_0 - \nu)^2 + (1/4\pi\tau_c)^2}\right]\,, \tag{4.30}$$

which is plotted in Figure . In the limit $\tau_c \to \infty$ the spectrum has the character of delta function centered at ν_0. In this limit the mode spectrum is a sharp spike (line) at ν_0. For finite τ_c, the power in a mode is distributed over a band of fequencies centered on mode frequency ν_0. Using FWHM to define the width $\Delta\nu_c$ of the power spectrum we find that

$$\Delta\nu_c = \frac{1}{2\pi\tau_c} = \frac{c}{2nL}\frac{\mathcal{L}}{2\pi} = \frac{\omega_0}{2\pi Q} = \frac{\nu_0}{Q} \equiv \frac{\gamma}{\pi}\,. \tag{4.31}$$

Thus the cavity mode acquires a width ν_0/Q known as the cavity line width. [2] Large values of Q imply sharply defined mode frequencies relative to the optical frequency. Another quality factor $\mathcal{F}$, called the finesse of the cavity, is used to characterize the sharpness of cavity modes. cavity finesse is defined to be the ratio of the longitudinal mode spacing $\Delta\nu_{ax} = c/2nL$ to the cavity linewidth $\Delta\nu_c$

$$\mathcal{F} = \frac{\Delta\nu_{ax}}{\Delta\nu_c} = \Delta\nu_{ax} 2\pi\tau_c = \frac{c}{2nL} 2\pi \frac{2nL}{c\mathcal{L}} = \frac{2\pi}{\mathcal{L}} \,. \tag{4.32}$$

Cavity finesse gives an indication of the sharpness of mode frequencies relative to the frequency spacing of cavity modes. For 90 cm long cavity with 4% loss per pass we find

$$\Delta\nu_c = \frac{3.00 \times 10^8}{2 \times 0.90} \times \frac{0.04}{2\pi} \,\mathrm{Hz} = 1.06\,\mathrm{MHz} \approx 1.1\,\mathrm{MHz} \,. \tag{4.33}$$

The axial mode separation is

$$\Delta\nu_{ax} = \frac{c}{2nL} = \frac{3.00 \times 10^8}{2 \times 0.90} \,\mathrm{MHz} = 167\,\mathrm{MHz} \tag{4.34}$$

Thus $\Delta\nu_c$ is small compared to the axial mode spacing.

[2] It is important to realize that widths are defined differently by different authors and one has to be careful in interpreting what the authors mean when they use the term linewidth.

4.2 Resonance Properties of Laser Resonators

Optical resonators can also be used as spectrum analyzers, filters or as build up cavities. For these applications the resonator mirrors must be patially transmitting to permit coupling of external fields to the cavity.

Consider a resonator filled with a dielectric of refractive index n and absorption coefficient α. A monochromatic field $\mathcal{E}_i$ is incident on mirror M_1. We assume that the incident field is spatially mode matched to one of the cavity modes. Let $\mathcal{E}_{_R}$ and $\mathcal{E}_{_T}$ denote the reflected and transmitted fields and $\mathcal{E}_+$ and $\mathcal{E}_-$ the right and left going fields inside the resonator. All fields have the dominant time dependence $e^{-i\omega t}$ and dominant space dependence $e^{ikz-\alpha z/2}$. Then in the steady-state the fields must satisfy the boundary conditions at $z = 0$ and $z = L$

$$\mathcal{E}_+(0) = \sqrt{\mathcal{T}_1}\mathcal{E}_i(0) + \sqrt{\mathcal{R}_1}\mathcal{E}_-(0) \tag{4.35a}$$

$$\mathcal{E}_{_R}(0) = -\sqrt{\mathcal{R}_1}\mathcal{E}_i(0) + \sqrt{\mathcal{T}_1}\mathcal{E}_-(0) \tag{4.35b}$$

$$\mathcal{E}_-(L) = \sqrt{\mathcal{R}_2}\mathcal{E}_+(L) = \sqrt{\mathcal{R}_2}\mathcal{E}_+(0)e^{-\alpha L/2+i(kL+\Psi)} \tag{4.35c}$$

$$\mathcal{E}_{_T}(L) = \sqrt{\mathcal{T}_2}\mathcal{E}_+(L) = \sqrt{\mathcal{T}_2}\mathcal{E}_+(0)e^{-\alpha L+i(kL+\Psi)} \tag{4.35d}$$

Here Ψ represents phase changes, other than kz, experienced by the wave due to propagation. It may includes, for example, the Guoy phase shift and any other contributions due to the transverse spatial structure of the wave.

Since $\mathcal{E}_+$ and $\mathcal{E}_-$ are part of the field circulating inside the resonator it follows that

$$\mathcal{E}_-(0) = \sqrt{\mathcal{R}_2}\mathcal{E}_+(L)e^{-\alpha L/2+i\phi} = \sqrt{\mathcal{R}_2}\mathcal{E}_+(0)e^{-\alpha L+i2\phi}\,. \tag{4.36}$$

where the phase ϕ is related to the wave frequency $\omega = 2\pi\nu$ and the refractive index n of the medium filling the resonator by

$$\phi \equiv kL + \Psi = \frac{n\omega L}{c} + \Psi\,. \tag{4.37}$$

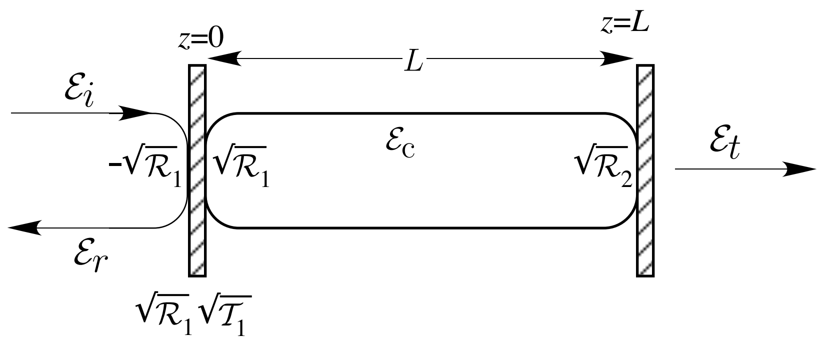

Figure 4.1: $\mathcal{E}_+$ and $\mathcal{E}_-$ are the right and left traveling part of the same circulating field $\mathcal{E}_c$.

With the help of Eq.(4.36) we can solve Eqs.(4.35) for $\mathcal{E}_+(0) \equiv \mathcal{E}_c$, $\mathcal{E}_{_T}(L) \equiv \mathcal{E}_{_T}$, and $\mathcal{E}_{_R}(0) \equiv \mathcal{E}_{_R}$ to find

$$\mathcal{E}_c = \frac{\sqrt{\mathcal{T}_1}}{1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}e^{i2\phi}}\mathcal{E}_i(0) \tag{4.38a}$$

$$\mathcal{E}_{_T} = \frac{\sqrt{\mathcal{T}_1\mathcal{T}_2}e^{-\alpha L}e^{i\phi}}{1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}e^{i2\phi}}\mathcal{E}_i(0)\,, \tag{4.38b}$$

$$\mathcal{E}_{_R} = \frac{\sqrt{\mathcal{R}_1}-\sqrt{\mathcal{R}_2}e^{-\alpha L+i2\phi}}{1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}e^{i2\phi}}\mathcal{E}_i(0) \tag{4.38c}$$

From these equations, we find that the ratios of the transmitted, reflected, and intracavity intensities to the intensity of the incident wave can be written as

$$\mathcal{T}_c \equiv \frac{I_{_T}}{I_0} = \frac{\mathcal{T}_{\rm cmax}}{1+F\sin^2\phi}\,, \tag{4.39a}$$

$$\mathcal{R}_c \equiv \frac{I_{_R}}{I_0} = \frac{\mathcal{R}_{\rm cmin}+F\sin^2\phi}{1+F\sin^2\phi}\,, \tag{4.39b}$$

$$\mathcal{I}_c \equiv \frac{I_{_c}}{I_0} = \frac{\mathcal{I}_{\rm cmax}}{1+F\sin^2\phi}\,, \tag{4.39c}$$

where we have introduced the following definitions

$$F = \frac{4\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}}{(1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2}\,, \tag{4.40a}$$

$$\mathcal{T}_{\rm cmax} = \frac{\mathcal{T}_1\mathcal{T}_2\, e^{-\alpha L}}{(1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2}\,, \tag{4.40b}$$

$$\mathcal{R}_{\rm cmin} = \frac{(\sqrt{\mathcal{R}_1}-\sqrt{\mathcal{R}_2}e^{-\alpha L})^2}{(1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2}\,, \tag{4.40c}$$

$$\mathcal{I}_{\rm cmax} = \frac{\mathcal{T}_1}{(1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2}\,. \tag{4.40d}$$

Here $\mathcal{T}_c$ and $\mathcal{R}_c$ represent, respectively, the power transmission and reflection coefficients of the cavity and $\mathcal{I}_c$ represents the factor by which the intracavity intensity is enhanced compared to the incident intensity. The significance of various quantities introduced in Eqs.(4.40) will become clear in the following paragraphs. Equations (4.39) indicate that $\mathcal{T}_c$, $\mathcal{R}_c$, and $\mathcal{I}_c$ are periodic functions of ϕ. They pass through a series of maxima and minima as ϕ varies.

A plot of cavity transmission and reflection as a function of ϕ is shown in Figs. (4.2) and (4.3). It attains a maximum value $\mathcal{T}_{\rm cmax}$ given by Eq.(4.40b) for

$$\phi \equiv \frac{n\omega L}{c} + \Psi = p\pi\,, \tag{4.41}$$

where p is an integer. The maxima are separated by $\phi\phi = \pi$. We also note from Fig. (4.2) that even when each mirror by itself may have a small transmission, cavity as a whole may have much larger transmission. This is a dramatic consequence of wave interference. This can be seen by considering Fig. (**??**) where the transmitted field is written as a superposition of partial transmissions as the wave injected into the cavity bounces back and forth between the mirrors. With this interpretation the integer p is referred to as the order of a transmission maximum and serves as an index to label it.

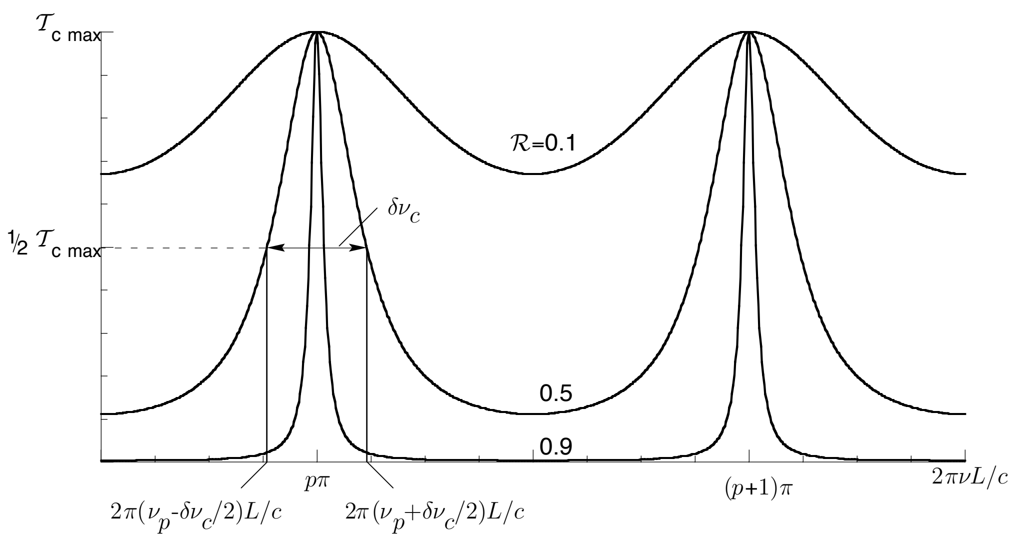

Figure 4.2: Transmission from a Fabry-Perot cavity as a function of ϕ.

Half way between two consecutive maxima, the transmission has a minimum given by

$$\mathcal{T}_{\rm cmin} = \frac{\mathcal{T}_{\rm cmax}}{1+F}\,. \tag{4.42}$$

These minima are located at

$$\phi \equiv \frac{n\omega L}{c} + \Psi = \left(p+\frac{1}{2}\right)\pi \tag{4.43}$$

The condition for maximum transmission (4.41) implies that maximum transmission is attained when the frequency of the incident field coincides with the frequencies determined by

$$\nu_p = \frac{c}{2nL}\left(p - \frac{\Psi}{\pi}\right)\,. \tag{4.44}$$

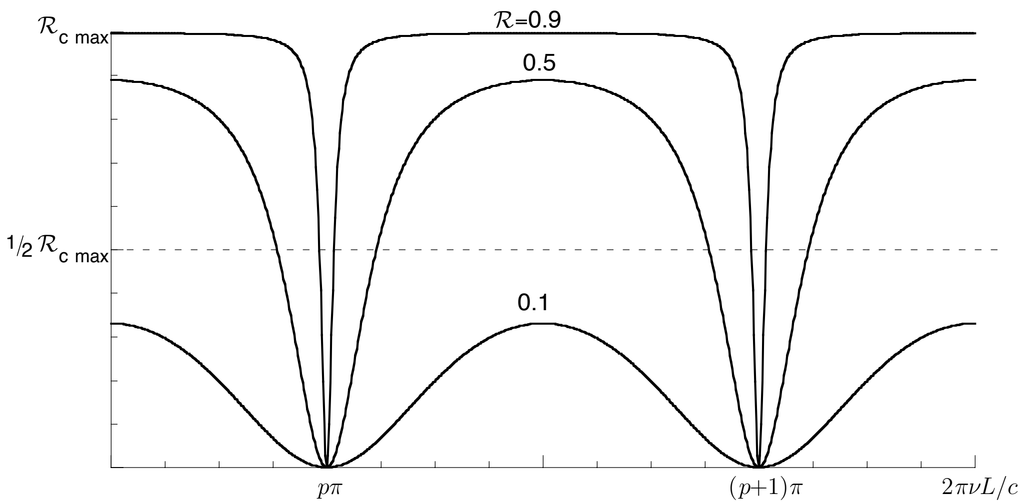

Figure 4.3: Reflection from a Fabry-Perot cavity as a function of ϕ.

These frequencies are precisely the cavity mode frequencies given by Eq.(**??**). It follows that the cavity transmission is a maxium whenever the incident field frequency is in resonance with a cavity mode frequency.

Assuming that the incident field is spatially mode matched to a particular transverse mode, the frequency interval between successive maxima will be

$$\nu_{p+1} - \nu_p = \frac{c}{2nL} \equiv \Delta\nu_{ax} \, , \tag{4.45}$$

which is equal to the axial mode separation for the cavity. In the context of resonators as spectrum analyzers, this interval is known as the free spectral range

$$\Delta\nu_{_{FSR}} = \frac{c}{2nL} \, . \tag{4.46}$$

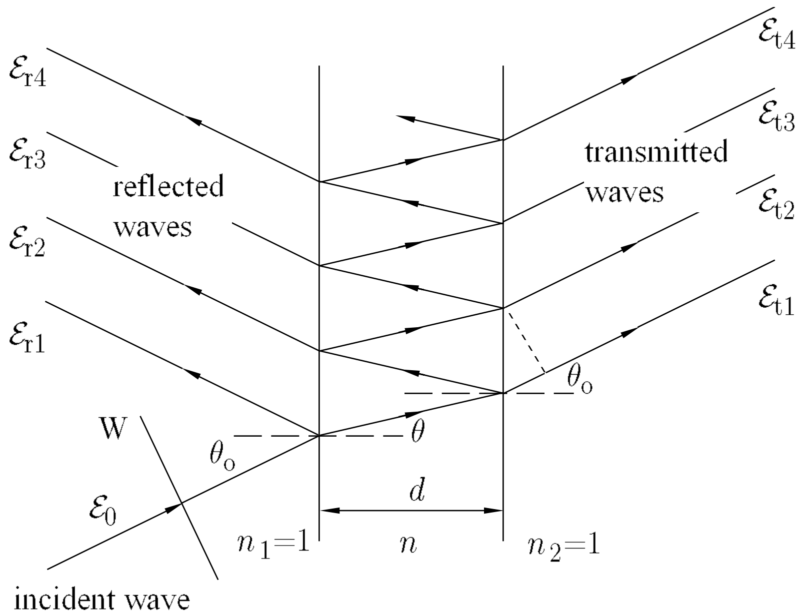

Figure 4.4: The reflected and transmitted beams can be written as a superposition of multiple beams generated by partial transmission and reflection of the beam entering the Fabry-Perot.

From Fig. (4.2), we note that the sharpness of transmission peaks increases with increasing coefficient of finesse F, which is controlled by the mirror reflectivities $\mathcal{R}_1$, $\mathcal{R}_2$, and internal losses $2\alpha L$, in addition to the quality of the resonators construction. When the transmission and internal losses $\mathcal{T}_1 = 1 - \mathcal{R}_1$, $\mathcal{T}_2 = 1 - \mathcal{R}_2$, and $\mathcal{L}_i = 1 - e^{2\alpha L}$ are small, we can use the much simpler expression for the coefficient of finesse

$$\begin{aligned} F &\equiv \frac{4\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}}{(1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2} \\ &\approx \frac{8[2-(\mathcal{T}_1+\mathcal{T}_2+2\alpha L)]}{(\mathcal{T}_1+\mathcal{T}_2+2\alpha L)^2} = \frac{8[2-\mathcal{L}]}{\mathcal{L}^2} \approx \frac{16}{\mathcal{L}^2}\,. \end{aligned} \tag{4.47}$$

The last approximation is good within a few percent as long as the total loss $\mathcal{L} = \mathcal{L}_T + \mathcal{L}_I \leq 0.1$.

A measure of the sharpness of the transmission maxima is their

FWHM (full width at half maximum). Let us consider a transmission maximum at $\nu = \nu_p$ and denote its FWHM by $\Delta\nu_c$. Then the frequency $\nu_p + \Delta\nu_c/2$ where the transmission falls to half of its maximum value at ν_p is determined by $\mathcal{T}_c(\nu_p + \frac{1}{2}\Delta\nu_c) = \frac{1}{2}\mathcal{T}_c(\nu_p)$. With the help of Eq. (4.56a) this leads to

$$\frac{\mathcal{T}_{\mathrm{cmax}}}{1 + F\sin^2(p\pi + \pi\Delta\nu_c nL/c)} = \frac{1}{2}\mathcal{T}_{\mathrm{cmax}} \tag{4.48}$$

or

$$F\sin^2(\Delta\nu_c \pi nL/c) = 1 \tag{4.49}$$

where we have used the result $\sin^2(p\pi + \theta) = \sin^2\theta$. Solving this we find that $\Delta\nu_c$ is determined by

$$\sin(\Delta\nu_c \pi nL/c) = \frac{1}{\sqrt{F}} \tag{4.50}$$

Here the negative root has been ignored because, by definition, $\Delta\nu_c$ is positive.

If the coefficient of finesse F is large, the argument of the sine function is small. We can then use the small angle approximation $\sin\theta \approx \theta$ and obtain

$$\Delta\nu_c = \frac{2}{\pi\sqrt{F}} \cdot \frac{c}{2nL} \equiv \frac{\Delta\nu_{_{FSR}}}{\mathcal{F}} \tag{4.51}$$

where $\Delta\nu_{_{FSR}} = c/2nL$ is the free spectral range and $\mathcal{F}$ is the finesse of the resonator given by

$$\mathcal{F} = \frac{\pi\sqrt{F}}{2} = \frac{\pi(\mathcal{R}_1\mathcal{R}_2)^{1/4}e^{-\alpha L/2}}{1 - \sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}} \tag{4.52}$$

It is clear that the width $\Delta\nu_c$ of a transmission peak is a small fraction of the separation between successive transmission peaks (free spectral range) when the resonator finesse $\mathcal{F}$ is large. When a resonator is used as a spectrum analyzer, its ability to resolve two closely spaced spectral frequency in the input signal is limited by the width $\Delta\nu_c$. Large finesse $\mathcal{F}$ is a measure of the resolving power of a resonator. By writing the finesse as the ratio

$$\mathcal{F} = \frac{\Delta\nu_{_{FSR}}}{\Delta\nu_c} = \frac{2\pi\tau_c}{\tau_{_R}} \tag{4.53}$$

we see that $\mathcal{F}$ is the number of transmission peaks that can fit in one free-spectral range. Also using the relation $\Delta\nu_c = 1/2\pi\tau_c$ and $\Delta\nu_{_{FSR}} = 1/\tau_{_R}$ we see that the finesse is 2π times the number of round trips that the wave makes in one cavity life time. For small losses, the expression for the finesse simplifies to

$$\mathcal{F} = \frac{2\pi}{\mathcal{L}} = \frac{2\pi}{\mathcal{L}_{_T} + \mathcal{L}_{_I}} \tag{4.54}$$

Using this, this expression for the finesse the width can be written as

$$\Delta\nu_c = \frac{c\mathcal{L}}{2nL} \equiv \frac{\mathcal{L}}{\tau_{_R}} \tag{4.55}$$

Thus $\Delta\nu_c$ is the fractional power loss per roundtrip.

In terms of $\mathcal{F}$, we can rewrite the transmission and reflection coefficients and the build up factor as

$$\mathcal{T}_c = \frac{\mathcal{T}_{\text{cmax}}}{1 + (2\mathcal{F}/\pi)^2 \sin^2\phi}\,, \tag{4.56a}$$

$$\mathcal{R}_c = \frac{\mathcal{R}_{\text{cmin}} + (2\mathcal{F}/\pi)^2 \sin^2\phi}{1 + (2\mathcal{F}/\pi)^2 \sin^2\phi}\,, \tag{4.56b}$$

$$\mathcal{I}_c = \frac{\mathcal{I}_{\text{cmax}}}{1 + (2\mathcal{F}/\pi)^2 \sin^2\phi}\,, \tag{4.56c}$$

In the neighborhood of a transmission peak at $\nu_p = p\Delta\nu_{_{FSR}}$ we can write

$$\nu = \nu_p + (\nu - \nu_p)\,. \tag{4.57}$$

Then, for a high-finesse cavity near a transmission peak

$$\begin{aligned}
\mathcal{T}_c &= \frac{\mathcal{T}_{\text{cmax}}}{1+(2\mathcal{F}/\pi)^2\sin^2[\pi(\nu-\nu_p)/\Delta\nu_{_{FSR}}]} \\
&\approx \frac{\mathcal{T}_{\text{cmax}}}{1+(2\mathcal{F}/\pi)^2[\pi(\nu-\nu_p)/\Delta\nu_{_{FSR}}]^2} \\
&= \frac{\mathcal{T}_{\text{cmax}}}{1+[2(\nu-\nu_p)/\Delta\nu_c]^2} \\
&= \frac{\mathcal{T}_{\text{cmax}}(\Delta\nu_c/2)^2}{(\nu-\nu_p)^2+(\Delta\nu_c/2)^2} = \frac{\mathcal{T}_{\text{cmax}}\pi\Delta\nu_c}{2}\cdot\frac{1}{\pi}\cdot\frac{\Delta\nu_c/2}{(\nu-\nu_p)^2+(\Delta\nu_c/2)^2} \\
&\equiv \frac{\mathcal{T}_{\text{cmax}}\pi\Delta\nu_c}{2}S(\nu)
\end{aligned} \tag{4.58}$$

Here the function $S(\nu)$ is the line shape function. This form is particularly useful when a high-finesse resonator is used as a filter.

We have derived three different expressions for $\Delta\nu_c$

$$\Delta\nu_c = \begin{cases} \frac{c}{2nL}\frac{1}{\pi}\cdot\frac{1-\sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L}}{(\mathcal{R}_1\mathcal{R}_2)^{1/4}e^{-\alpha L/2}} \equiv \frac{c}{2nL\mathcal{F}} \\ \frac{c}{2nL}\frac{1}{2\pi}(2\alpha L - \ln\mathcal{R}_1\mathcal{R}_2) \equiv \frac{\nu}{Q} \\ \frac{c}{2nL}\frac{1}{2\pi}(2\alpha L + \mathcal{T}_1 + \mathcal{T}_2) \end{cases} \tag{4.59}$$

In practice, the first and second expressions for $\Delta\nu_c$ lead to the same answer within a few percent for $\mathcal{R}_1$ and $\mathcal{R}_2$ as large as 0.5. Furthermore, both the first and second expressions give the same result as the third expression for $\mathcal{R}_1$, $\mathcal{R}_2$, and $1-e^{-2\alpha L} \leq 0.1$.

Case I: $\mathcal{R}_1 \approx \mathcal{R}_2 = 0.98$, $2\alpha L \approx 0$

$$\Delta\nu_c = \begin{cases} \frac{c}{2nL} \times \frac{1}{\pi} \times 0.02020305 = \frac{c}{2nL} \times 6.4308 \times 10^{-3} \\ \\ \frac{c}{2nL} \times \frac{1}{\pi} \times 0.020202071 = \frac{c}{2nL} \times 6.4307 \times 10^{-3} \\ \\ \frac{c}{2nL} \times \frac{1}{\pi} \times 0.02 = \frac{c}{2nL} \times \times 6.3662 \times 10^{-3} \end{cases}$$

Case II: $\mathcal{R}_1 \approx \mathcal{R}_2 = 0.5$, $2\alpha L \approx 0$

$$\Delta\nu_c = \begin{cases} \frac{c}{2nL} \times \frac{1}{\pi} \times 0.7071068 = \frac{c}{2nL} \times 0.225 \\ \\ \frac{c}{2nL} \times \frac{1}{\pi} \times 0.69314718 = \frac{c}{2nL} \times 0.221 \\ \\ \frac{c}{2nL} \times \frac{1}{\pi} \times 0.5 = \frac{c}{2nL} \times 0.159 \end{cases}$$

For $L = 1.00$ m in air, $\mathcal{R}_2 = \mathcal{R}_2 \approx 0.98$ and $\alpha L = 0$, we obtain the cavity linewidth to be

$$\varDelta\nu_c = \frac{c}{2L}\frac{1}{2\pi}(\mathcal{T}_1 + \mathcal{T}_2) = \frac{3.00 \times 10^8}{2 \times 1.00 \times 2\pi} \times 0.04 = 9.55 \times 10^5\,\text{Hz} \approx 1.0\,\text{MHz}$$

The finesse is found to be

$$\mathcal{F} = \frac{c}{2L} = \frac{2\pi}{\mathcal{L}} = 157$$

Cavity Buildup

The intracavity field build up is also a periodic function of ϕ and has a maximum on transmission resonance

$$\mathcal{I}_{\text{cmax}} = \frac{\mathcal{T}_1}{(1 - \sqrt{\mathcal{R}_1\mathcal{R}_2}e^{-\alpha L})^2} \tag{4.60}$$

For a high-finesse cavity $\mathcal{R}_1, \mathcal{R}_2 \approx 1$ and $2\alpha L \ll 1$

$$\mathcal{I}_{\text{cmax}} = \frac{4\mathcal{T}_1}{(\mathcal{T}_1 + \mathcal{T}_2 + 2\alpha L)^2} \tag{4.61}$$

If $\mathcal{T}_1 \approx \mathcal{T}_2 + 2\alpha L$, which is a condition for zero reflection from the cavity we have a maximum buildup factor

$$\mathcal{I}_{\text{cmax}} = \frac{1}{\mathcal{T}_1} \tag{4.62}$$

If $\mathcal{T}_2 + 2\alpha L \ll \mathcal{T}_1$ we have the largest buildup factor

$$\mathcal{I}_{\text{cmax}} = \frac{4}{\mathcal{T}_1} \tag{4.63}$$

In practice, even modest buildup factors, like 10, give a 100-fold increase in nonlinear optical experiments, such as second harmonic generation.