8

Quantum Statistics of Laser Light

There is another profound change that taks place near threshold. This is in the transformation of laser photon statistics. In fact almost all properties of laser light, including directionality, linewidth, intensity, and coherence, can in principle, be attained by light from natural sources. Ultimately, its the statistics of photons that distinguishes it from thermal sources.

8.1 Equations of Motion

We have derived the following equations for the photon number inside the cavity and the population inversion

$$\dot{q} = Knq - \gamma_c q \tag{8.1a}$$

$$\dot{n} = r_p - \gamma_2 n - Knq \tag{8.1b}$$

These equations do not take into account spontaneous emission of radiation into the cavity mode. This can be seen by noting that the rate of emission represented by the first term on the rhs of Eq.(1.1) is proportional to q. It represents emission induced by the field. In practice, excited atoms will decay spontaneously by emitting a photon even if no quanta are present in the cavity. This effect can be taken into account by replacing q by $q+1$ in the emission and absorption terms. With this modification the equations become

$$\dot{q} = Kn(q+1) - \gamma_c q \tag{8.2a}$$

$$\dot{n} = r_p - \gamma_2 n - Kn(q+1) \tag{8.2b}$$

If atomic dynamics are fast compared to the field dynamics, atoms can reach equilibrium rapidly and follow the field adiabatically. We can then solve for n

$$n = \frac{r_p/\gamma_2}{1+(q+1)K/\gamma_2} = \frac{n^{(0)}}{1+(q+1)/q_s} \tag{8.3}$$

where the unsaturated inversion $n^{(0)}$ and saturation photon number q_s are given by

$$n^{(0)} = \frac{r_p}{\gamma_2}, \qquad q_s = \frac{\gamma_2}{K}, \qquad r = \frac{n^{(0)}}{n_{th}} = \frac{Kr_p}{\gamma_2\gamma_c} \tag{8.4}$$

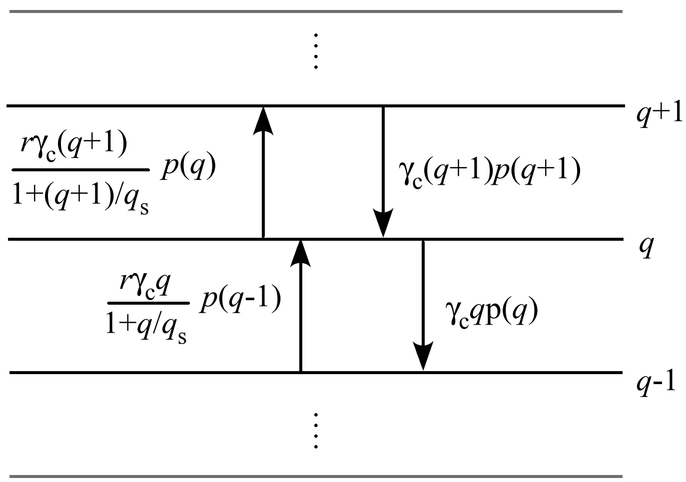

FIGURE 8.1
Transition rates between cavity photon number states with $q+1$, q and $q-1$ quanta.

Substituting these in the equation for q we obtain

$$\dot{q} = \frac{\gamma_c r(q+1)}{1+(q+1)/q_s} - \gamma_c q \tag{8.5}$$

This equation tells us that the photon number inside the cavity increases due to emission at the rate $\gamma_c r(q+1)/(1+(q+1)/q_s)$ and decreases due to loss of photons at a rate $\gamma_c q$. This means when there are q photons in the cavity the atoms emit at a rate $\gamma_c r(q+1)/(1+(q+1)/q_s)$.

8.2 Photon Number Probabilities

These equation allow us to write the equations for the probability $p(q)$ that the cavity has q photons at time t. This probability will decrease if an atom emits into the cavity mode so that there are $q+1$ photons. The rate of emission when there are q photons in the cavity is given by $\gamma_c r(q+1)/(1+(q+1)/q_s)$ multiplied by the probability $p(q)$. $p(q)$ will increase if there is emssion of a photon into the cavity containing initially $q-1$ photons. Similarly, $p(q)$ will increase if cavity loses a photons starting with $q+1$ photons. It will decrease if it loses a photon starting with q photons initially. We assume that atoms make their contributions independently, and also photons leave the cavity independently. We then find these probabilities are connected via [see Fig. (8.1)]

$$\begin{aligned}\dot{p}(q) = &-\frac{\gamma_c r(q+1)}{1+(q+1)/q_s}p(q) \quad + \quad \gamma_c(q+1)p(q+1) \\ &\begin{Bmatrix}\text{rate of decrease due}\\ \text{to emission of a}\\ \text{quantum into the}\\ \text{cavity with } q \text{ quanta}\end{Bmatrix} \quad \begin{Bmatrix}\text{rate of increase due}\\ \text{to loss of photons}\\ \text{from the cavity with}\\ q+1 \text{ quanta}\end{Bmatrix} \\ &+ \quad \frac{\gamma_c r q}{1+q/q_s}p(q-1) \quad - \quad \gamma_c q p(q) \\ &\begin{Bmatrix}\text{rate of increase due}\\ \text{to emission of a}\\ \text{quantum into the}\\ \text{cavity with } q-1\\ \text{quanta}\end{Bmatrix} \quad \begin{Bmatrix}\text{rate of decrease due}\\ \text{to loss of photons}\\ \text{from the cavity with}\\ q \text{ quanta}\end{Bmatrix}\end{aligned} \tag{8.6}$$

In the steady-state the up and down transitions rates between any two states should balance. Considering the transitions between cavity states with q and $q+1$, we find the corresponding probabilities $p(q)$ and $p(q-1)$are related by

$$\gamma_c q p(q) = \frac{\gamma_c r q}{1+q/q_s}p(q-1) \tag{8.7}$$

By iterating this equation, the steady-state photon number distribution $p(q)$ can be expressed in terms of $p(0)$ as

$$p(q) = p(0)r^q \prod_{k=1}^{q} \frac{1}{1+k/q_s} = \mathcal{C}\frac{(rq_s)^q}{(q_s+q)!}\,, \tag{8.8}$$

where the normalization constant $\mathcal{C}$ is determined by the requirement

$$\sum_{q=0}^{\infty} p(q) = 1\,. \tag{8.9}$$

Note that the factor r^q decreases monotonically with increasing q below threshold ($r < 1$), whereas above threshold ($r < 1$) it increases monotonically with icreasing q. The second factor $\frac{q_s^q}{(q_s+q)!}$ is always a decreasing function of q. Thus the distribution $p(q)$ is expected to have different behavior below and above threshold. Let us consider its behavior below and above tyhreshold in some detail.

8.2.1 Case I: Below threshold $r < 1$

In this case r^q decreases monotonically with q. Since the second factor is always a decreasing factor, overall $p(q)$ is a monotonicaly decreasing function of q. Hence for photon numbers q that have significant probabilities, the probability $p(q)$ can be written as

$$p(q) \approx C \times r^q \tag{8.10}$$

The constant C is determined by the normalization condition (8.9),

$$\sum_{q=0}^{\infty} p(q) = 1$$

$$\text{or} \qquad C\sum_{q=0}^{\infty} r^q = 1$$

$$\text{or} \qquad C\frac{1}{1-r} = 1$$

$$\text{or} \qquad C = r - 1$$

Using this expression for the normalization constant we can write the photon number distribution $p(q)$ below threshold as

$$p(q) = r^q(1-r). \tag{8.11}$$

This distribution depends only on pump ratio r. Using the relation between the pump ratio r and the mean photon number

$$\langle q \rangle = \sum_{q=0}^{\infty} qp(q) = \frac{r}{1-r}\,. \tag{8.12}$$

we can express the pump ratio r in terms of $\langle q \rangle$

$$r = \frac{\langle q \rangle}{\langle q \rangle + 1}. \tag{8.13}$$

We can then, finally, write the photon number distribution below threshold as

$$p(q) = \frac{< q >^q}{[1+ < q >]^{q+1}} \tag{8.14}$$

This is a monotonically decreasing function of q with $q = 0$ is being the most probable value where the probability distribution attains its maximum value. This distribution is known as the Bose Einstein distribution.

8.2.2 Case II: Far above threshold $r > 1$

In this case the photon number becomes large and we cannot ignore saturation. The photon number distribution can be written as

$$p(q) = \text{const} \times \frac{(rq_s)^q}{(q_s + q)!} \tag{8.15}$$

For sufficiently high intensities $q >> q_s$ for those values for which $p(q)$ is significant. In this case we have

$$p(q) \approx \text{const} \times \frac{(rq_s)^q}{q!}. \tag{8.16}$$

Once again this is a single parameter equation. The normailzation condition

$$\sum_{q=0}^{\infty} p(q) = 1 = \sum_{q=0}^{\infty} \frac{(rq_s)^q}{q!} = \text{const} \times e^{rq_s}. \tag{8.17}$$

Using this we can write the photon number distribution as

$$p(q) = \frac{(rq_s)^q}{q!} e^{-rq_s}. \tag{8.18}$$

In terms of the mean photon number

$$< q >= \sum_{q=0}^{\infty} qp(q) = e^{-rq_s} \sum_{q=0}^{\infty} q \frac{(rq_s)^q}{q!} = rq_s \tag{8.19}$$

we can write the photon number distribution as

$$p(q) = \frac{< q >^q}{q!} e^{-<q>} \tag{8.20}$$

which is the Poisson distribution.

8.2.3 Near threshold region: $|r-1| \sim 1/\sqrt{q_0}$

In this case we write the photon number distribution as

$$p(q) = p(0) \frac{(rq_s)^q}{(q_s + q)!} \tag{8.21}$$

Recall that in this case the mean photon number $< q >\sim \sqrt{q_s}$. The transition from $< q >\sim r/(1-r)$ to $< q >= (r-1)q_s$ occurs in a region $|r-1| \sim 1/\sqrt{q_s}$. In order to examine the threshold region more closely we introduce the pump parameter

$$a = (r-1)\sqrt{2q_s}.$$

In terms of a, laser threshold is $a = 0$. Above threshold $a > 0$ and below threshold $a < 0$. Near threshold region is then, say, $-10 < a < 10$. Then we can write

$$r = 1 + r - 1 = 1 + (r-1)\sqrt{2q_s} \times \frac{1}{\sqrt{2q_s}} \equiv 1 + \frac{a}{\sqrt{2q_s}} \approx e^{a/\sqrt{2q_s}}. \tag{8.22}$$

Since both $\sqrt{q_s}$ and q are large we can use Stirrling's approximation[1]

$$\begin{aligned}
\ln(q_s+q)! &= (q_s+q+1/2)\ln(q_s+q)-(q_s+q)-\frac{1}{2}\ln(2\pi)\\
&= (q_s+q+1/2)\left[\ln q_s+\ln(1+q/q_s)\right]-(q_s+q)-\frac{1}{2}\ln(2\pi)\\
&= \text{const}+q\ln q_s+(q_s+q+1/2)\left[\frac{q}{q_s}-\frac{1}{2}\frac{q^2}{q_s^2}+\frac{1}{3}\frac{q^3}{q_s^3}-\cdots\right]-q\\
&= \text{const}+q\ln q_s+q-\frac{1}{2}\frac{q^2}{q_s}+\frac{1}{3}\frac{q^3}{q_s^2}+\frac{q^2}{q_s}-\frac{1}{2}\frac{q^3}{q_s^2}+\frac{1}{2}\frac{q}{q_s}\cdots-q\\
&\approx \text{const}+q\ln q_s+\frac{1}{2}\frac{q^2}{q_s}
\end{aligned} \tag{8.23}$$

Using this result in Eq. (8.21) we find that to an excellent approximation $p(q)$ is given by

$$p(q)=\text{const}\times e^{q(\ln r+\ln q_s)-\ln(q_s+q)!}=\text{const}\times\exp\left[\frac{a}{\sqrt{2}}\frac{q}{\sqrt{q_s}}-\frac{1}{2}\frac{q^2}{q_s}\right]. \tag{8.24}$$

This shape of this distribution is shown in Fig. 8.2 for pump parameters below and above threshold in the region of threshold.

From this distribution we see that below threshold, $a<0$, it is an exponentially decreasing function of q

$$p(q)\approx\frac{|a|}{\sqrt{2q_s}}e^{-|a|q/\sqrt{2q_s}}. \tag{8.25}$$

Above threshold, $a>0$, it is a Gausssian distribution

$$p(q)=\text{const}\times\exp\left[-\frac{1}{2}\left(\frac{q}{\sqrt{q_s}}-\frac{a}{\sqrt{2}}\right)^2\right] \tag{8.26}$$

centered at $q=<q>=a\sqrt{q_s/2}=(r-1)q_s$ and constant width $<(\Delta q)^2>=q_s$. Note that the relative width $<(\Delta q)^2>/<q>^2=2/a^2$. From the steady-state photon number distribution $p(q)$ we obtain the mean photon number

$$\langle q\rangle=\sum_{q=0}^{\infty}qp(q)\to\int_0^{\infty}qp(q)dq=\sqrt{\frac{q_s}{2}}\left[a+\frac{2}{\sqrt{\pi}}\frac{e^{-\frac{1}{4}a^2}}{1+\text{erf}(a/2)}\right] \tag{8.27}$$

[1]Stirrling's approximation gives the factorial function as $m!=\sqrt{2\pi}m^{m+1/2}e^{-m}$ for $m\gg1$.

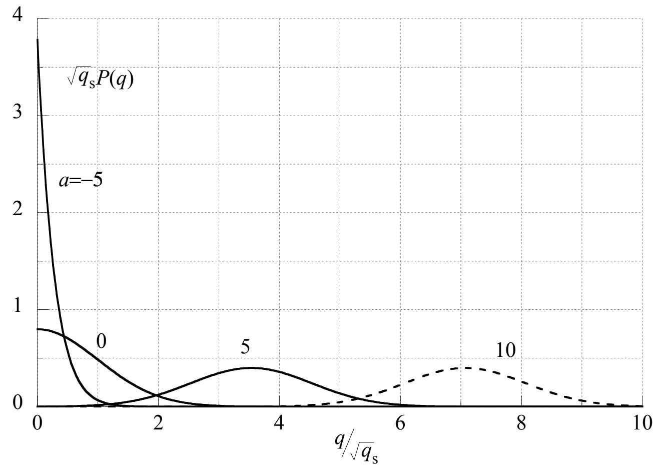

FIGURE 8.2
The cavity photon number distribution changes as the laser passes threshold. Below threshold it is peaked at $q = 0$ whereas above threshold it peaks at a nonzero value of q.

where $\text{erf}(x)$ is the error function of argument x.[2] In certain limits simpler expressions for the mean cavity photon number can be derived

$$\langle q \rangle = \begin{cases} \dfrac{\sqrt{2q_s}}{|a|} = \dfrac{1}{1-r} & a < 0,\ |a| \gg 1 \\ \sqrt{\dfrac{2q_s}{\pi}} \equiv q_{th} & a = 0 \\ \sqrt{\dfrac{q_s}{2}}\ a = q_s(r-1) & a \gg 1 \end{cases} \tag{8.28}$$

[2]The error function is defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy\, e^{-y^2}$. Some of its important propertise are

$$\begin{aligned} \text{erf}(x) &= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \cdots \right) \\ \text{erf}(-x) &= -\text{erf}(x) \\ \text{erf}(x) &\xrightarrow{|x| \gg 1} 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{2^2 x^4} - \cdots \right) \end{aligned}$$

The variance of q is

$$\langle(\Delta q)^2\rangle \equiv \int_0^\infty dq(q-\langle q\rangle)^2 p(q)$$
$$= q_s\left[1-\frac{a}{\sqrt{\pi}}\frac{e^{-\frac{1}{4}a^2}}{1+\mathrm{erf}(a/2)}-\frac{2}{\pi}\frac{e^{-a^2/2}}{[1+\mathrm{erf}(a/2)]^2}\right]. \qquad (8.29)$$

Using the small and large argument expansion of the error function, the following limiting forms of variance can be derived

$$\langle(\Delta q)^2\rangle = \begin{cases} \dfrac{2q_s}{a^2} & a<0,\ |a|\gg 1 \\ q_s\left(1-\dfrac{2}{\pi}\right) & a=0 \\ q_s & a\gg 1 \end{cases} \qquad (8.30)$$

We note that high above threshold the variance of the distribution stays constant at q_s with increasing a even while its mean $\langle q\rangle = q_s(r-1)$ increases. Thus in a relative sense the distribution narrows. Indeed the relative variance is given by

$$\frac{\langle(\Delta q)^2\rangle}{\langle q\rangle^2} = \begin{cases} 1 & a<0,\ |a|\gg 1 \\ \dfrac{\pi}{2}-1=0.57 & a=0 \\ 2/a^2 & a>5 \end{cases} \qquad (8.31)$$

From these expressions we see that below threshold the laser has large photon number fluctuations (of the same order as the mean) characteristic of thermal fields. As the laser approaches threshold relative photon number fluctuations decrease. Above threshold the relative photon number fluctuations rapidly die out and the laser approaches a state with negligible photon number flutuations.

The behavior of normalized photon number $< I >\equiv< q > \sqrt{2/q_s}$, its variance $\kappa_2 \equiv< \Delta q)^2 > / < q >^2$ and kurtosis $\kappa_3 \equiv< (\Delta q)^2 > / < q >^2$ is shown in Fig. 8.3 as a function of pump parameter a. Note that the transformation of laser light from an exponential photon number distribution below threshold to a gaussian above threshold happens in a very narrow range or pump parameter $|a| < 5$ corresponding to $|r-1| \leq 1/\sqrt{q_s}$. For most realistic laser systems, the saturation photon number $q_s \sim 10^8$. This means that the dramatic transformation of laser light from thermal state to coherent state is complete when the relative excess of gain over its threshold is less than a few one part in 10,000.

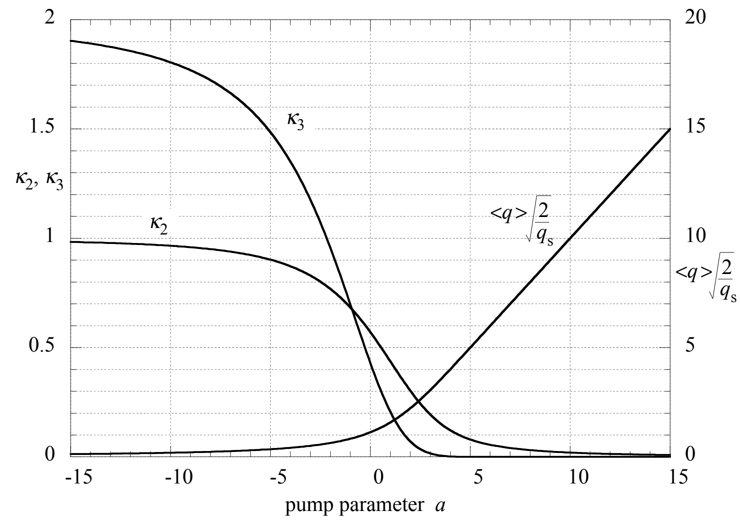

FIGURE 8.3
Behavior of mean photon number, normalized variance and kurtosis in the region of threshold.

8.3 Laser Linewidth

We have seen that in an empty cavity, the electric ield amplitude decays according to

$$\mathcal{E}(\boldsymbol{r}, t) = \mathcal{E}_0(\boldsymbol{r})\, e^{-t/2\tau_c - i\omega t + i\phi_0}\,. \tag{8.32}$$

This leads to a lorentzian spectral density with FWHM

$$\Delta\nu_c = \frac{1}{2\pi\tau_c} = \frac{\gamma_c}{2\pi} = \frac{1}{2\pi}\frac{c\mathcal{L}_c}{2nL}\,, \tag{8.33}$$

where $\mathcal{L}_c$ is the loss per cavity roundtrip.

When gain is present, cavity field amplitude does not decay. Does this mean the linewidth vanishes? This would be the case if spontaneous emission or losses were absent. Spontaneous emission of photons into the cavity mode results in contributions to the field amplitude with random phases. This gives the field a small spectral width which decreases as the cavity field amplitude becomes larger and well defined. We can understand this behavior of laser linewidth using a simple pictorial argument.

We have seen that the magnitude of the electric field amplitude increases rapidly in passing the threshold and stabilizes at a large and well defined value as $\langle(\Delta q)^2\rangle/\langle q\rangle^2 \to 0$ with increasing pump parameter. The phase, however, continues to perform a random walk due to spontaneous emission contribution.

This leads to the situation depicted in Fig. (8.4). We reprsent the electric field of the laser as a phasor of very large amplitude $\sqrt{\langle q \rangle}$. Its length is changed by one unit corresponding to a single spontaneously emitted photon into the cavity mode. The the phase change can be written as

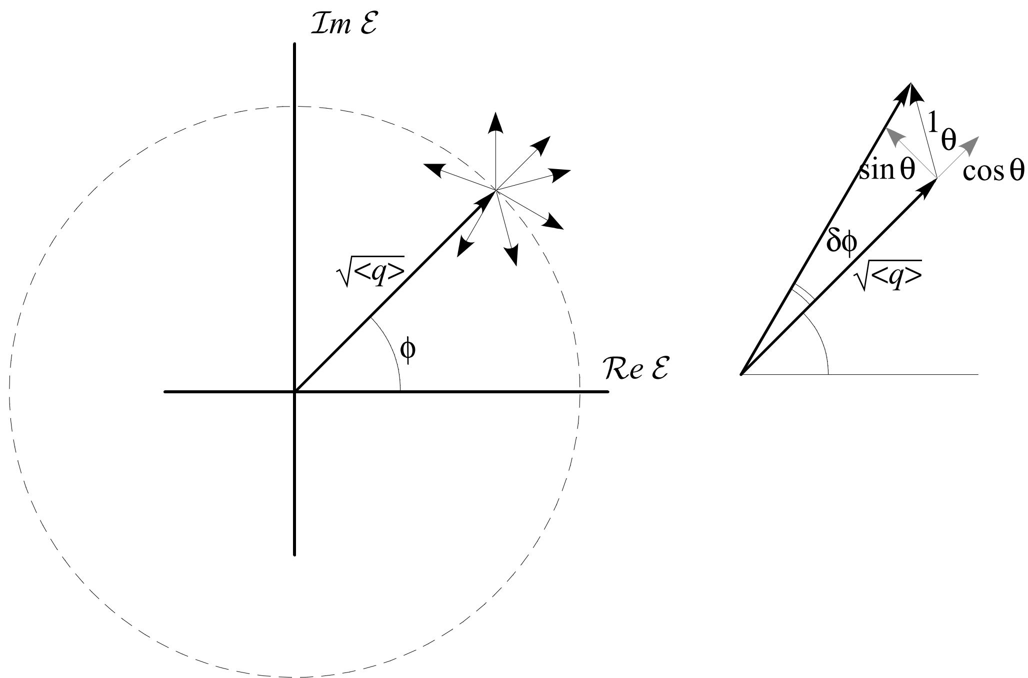

FIGURE 8.4
The effect of each spontaneous emission is to add a phasor of unit length with random phase θ to a phasor of large magnitude $\sqrt{\langle q \rangle}$ and phase ϕ.

$$\delta\phi = \frac{\sin\theta}{\sqrt{\langle q \rangle}} \tag{8.34}$$

$$\langle (\delta\phi)^2 \rangle = \frac{\langle \sin^2\theta \rangle}{\langle q \rangle} = \frac{1}{2\langle q \rangle} \, . \tag{8.35}$$

The rate at which spontaneous photons are emitted into the cavity mode is

$$R_s = \frac{N_2 \gamma_{21}}{p} \, , \tag{8.36}$$

where N_2 is the number of atoms in the upper lasing level, γ_{21} is the spontaneous decay rate of the upper level and p is the number of modes that are coupled to the atomic transition. This means that in time t, the phase diffuses by an amount

$$\langle(\Delta\phi)^2\rangle = \frac{1}{2\langle q\rangle}\frac{N_2\gamma_{21}}{p} \equiv \Delta\omega\, t\,. \tag{8.37}$$

This gives a spectral linewidth

$$\Delta\nu = \frac{\Delta\omega}{2\pi} = \frac{N_2\gamma_{21}}{4\pi\langle q\rangle p}\,. \tag{8.38}$$

We can understand how $\langle(\Delta\phi)^2\rangle$ translates into linewidth as follows. Write the complex electric field at time $t = 0$ as

$$\mathcal{E}(0) = \mathcal{E}_o(\boldsymbol{r})e^{i\phi_0}\,. \tag{8.39}$$

Then the electric field at a later time t will be

$$\mathcal{E}(t) = \mathcal{E}_o(\boldsymbol{r})e^{-i\omega t + i\phi(t)} \tag{8.40}$$

Apart from the uniform rotation of the electric field phasor implied by the factor $e^{-i\omega t}$, there is a random contribution to the rotation of the phasor due to the $e^{i\phi(t)}$ term. Since many spontaneous emissions contribute to this motion, its average contribution to the amplitude will be given by[3]

$$\mathcal{E}(t) = \mathcal{E}_0(\boldsymbol{r})e^{-i\omega t + i\phi_0}\langle e^{i(\phi(t)-\phi_0)}\rangle \equiv \mathcal{E}_0(\boldsymbol{r})e^{-i\omega t + i\phi_0}\langle e^{i\Delta\phi(t)}\rangle\,. \tag{8.41}$$

Here the averaging implied by the last term is over many spontaneous emissions and can be evaluated as follows

$$\langle e^{i\Delta\phi(t)}\rangle = \sum_{n=0}^{\infty}\frac{i^n\langle(\Delta\phi(t))^2\rangle}{n!}\,. \tag{8.42}$$

Since the phase change $\Delta\phi(t)$ is due to many independent emissions, we expect it to obey gaussian statistics. This means odd order moments of $\Delta\phi(t)$ will vanish and even order moments can be expressed in terms of the second order moment

$$\langle(\Delta\phi(t))^{2m+1}\rangle = 0\,. \tag{8.43}$$

$$\langle(\Delta\phi(t))^{2m}\rangle = (2m-1)!!\langle(\Delta\phi(t))^2\rangle^m\,. \tag{8.44}$$

Using these results we find

$$\langle e^{i\Delta\phi(t)}\rangle = \sum_{m=0}^{\infty}\frac{(-1)^m\langle(\Delta\phi(t))^2\rangle^m}{m!2^m} = e^{-\langle(\Delta\phi(t))^2\rangle/2} = e^{-\frac{1}{2\langle q\rangle}\frac{N_2\gamma_{21}}{p}\frac{t}{2}} \equiv e^{-t/2\tau}\,. \tag{8.45}$$

[3]Spontaneous emission also results in a random contribution to the length of the phasor.

This leads to a linewidth

$$\Delta\nu = \frac{1}{2\pi\tau} = \frac{n_2\gamma_{21}}{4\pi p\langle q\rangle} \tag{8.46}$$

Note that the field amplitude doesn't decay in time unlike the empty cavity case [see Fig. (8.4)]. It is only the average field amplitude that displays this characteristics due to phase diffusion. The field undergoes sinusoidal oscillations and every time an emission occurs its phase is interrupted. The effect of many such interruptions is that the coherent field amplitude decays in time.[4]

We can relate $\langle q\rangle$ to the average power emitted by the laser

$$P = \gamma_c\langle q\rangle\hbar\omega \tag{8.47}$$

and write the linewidth as

$$\Delta\nu = \frac{\gamma_c\hbar\omega}{4\pi P}\frac{N_2\gamma_{21}}{p} = \Delta\nu_c\frac{\hbar\omega}{2P}\frac{N_2\gamma_{21}}{p}\,. \tag{8.48}$$

The ratio γ_{21}/p spontaneous emission per mode can be expressed in terms of laser threshold inversion. recall that the threshold inversion was defined by

$$K(\Delta N)_{th} = \frac{\gamma_{21}(\Delta N)_{th}}{p} = \gamma_c\,. \tag{8.49}$$

Using this relation we find

$$\Delta\nu = \Delta\nu_c\frac{\hbar\omega}{2P}\frac{N_2\gamma_c}{(\Delta N)_{th}} = \Delta\nu_c\frac{\hbar\omega}{P}\frac{N_2}{(\Delta N)_{th}}\pi\frac{\gamma_c}{\pi} = \pi(\Delta\nu_c)^2\frac{\hbar\omega}{P}\frac{N_2}{(\Delta N)_{th}}\,. \tag{8.50}$$

Since P increases dramatically in passing laser threshold, laser linewidth alos changes (narrows) dramatically in passing the threshold of laser oscillations.

He:Ne laser

Using some typical values for a He:Ne laser ($\lambda = 633$ nm) $\mathcal{R}_1 \approx 1.0$, $\mathcal{R}_2 \approx 0.97$, $L = 30$ cm, and $P = 1$ mW, we find that its linewidth $\Delta\nu = 0.07$ Hz.

GaAs/GaAlAs Lasers

For these lasers with $\nu = 3.53\times10^{14}$ Hz ($\lambda = 850$ nm), $L = 3\times10^{-2}$ cm, $P = 1$ mW, $R = 31$ % (Fresnel reflection), $n = 3.5$, $\Delta\nu_c = (1-\mathcal{R})c/2nL \approx 3\times10^{10}$ Hz we find $\Delta\nu = 4.5\times10^6$ Hz.

These linewidths are way too small compared to the observed values. In real life the linewidth is limited not by spontaneous emission but by mechanical and environmental perturbations.

[4]The situation is similar to the decay of an atom which undergoes sinusoidal Rabi oscillations when interacting with a driving field. Every time an emission occurs its phase is interrupted. The time average effect of many such emissions leads to a decay of the coherent amplitude at rate determined by the rate of interruptions.

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