10

Mode Locking

1. A. Siegman, *Lasers* [University Science Books, Sausalito, CA, 1986], Chapter 27

2. A. Yariv, *Quantum Electronic* [Wiley, New York, 1989], Chapter 20

3. O. Svelto, *Principles of Lasers* translated by D. C. Hanna [Plenum Press, New York, NY, 1998], Chapter 8.

10.1 Mode Locked Pulses

Consider a laser with a large number $N \gg 1$ of oscillating axial modes having the same polarization. If $\Delta\nu_{osc}$ is the oscillation bandwidth (frequency range over which gain exceeds loss) and $\delta\nu_{ax}$ is the axial mode separation, the number of oscillating modes N is given by

$$N = \frac{\Delta\nu_{osc}}{\Delta\nu_{ax}} \tag{10.1}$$

For simplicity asssume that all oscillating modes have nearly the same amplitude and that N is an odd integer so that $(N-1)/2$ is also integer. Recall that axial mode frequencies are separated by $\Delta\nu_{ax} = c/2L = 1/\tau_R$. If $\nu_0 = Mc/2L$ (M is a number $\sim 10^7$) is central mode frequency (average frequency), then the frequencies and wave numbers for the modes can be written as

$$\left.\begin{aligned} &\nu_r = \nu_0 + r\Delta\nu_{ax} \\ &k_r \equiv \frac{2\pi\nu_r}{c} = \frac{2\pi\nu_0}{c} + \frac{2\pi r\Delta\nu_{ax}}{c} \end{aligned}\right\} \; -\frac{(N-1)}{2} \le r \le \frac{(N-1)}{2} \tag{10.2}$$

The electric field of the laser can then be written as the superposition of standing wave fields of different modes

$$\mathcal{E}(z,t) = i\mathcal{E}_0 \sum_{r=-(N-1)/2}^{(N-1)/2} e^{-i2\pi\nu_r t + i\varphi_r} \sin k_r z$$

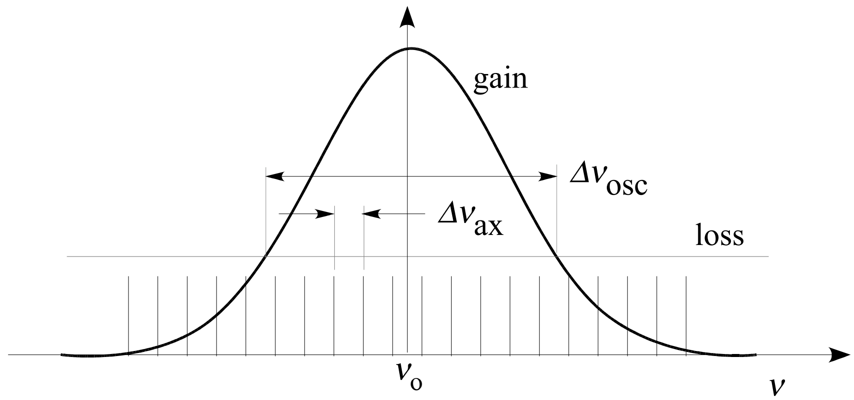

FIGURE 10.1
Number of oscilating modes N depends on the nature (homogeneous/inhomogeneous) of gain medium, spatial distribution of gain and loss, axial mode separation and relative excess of gain over loss.

where φ_r is the initial phase of the rth mode. In general φ_r varies randomly from mode to mode. For phase locked modes[1] the relation $\varphi_{r+1} - \varphi_r = \varphi_0$ holds, where φ_0 is a constant. In what follows we will choose $\varphi_0 = 0$ for convenience. This assumption does not limit the generality of the conclusions reached.

By writing $\sin(k_r z) = (e^{ik_r z} - e^{ik_r z})/2i$, we can express the standing wave electric field of each mode as the superposition of left and right traveling waves

$$\begin{aligned}\mathcal{E}(z,t) &= \frac{\mathcal{E}_0}{2}\sum_{r=-(N-1)/2}^{(N-1)/2}\left[e^{-i2\pi\nu_r(t-z/c)} - e^{-i2\pi\nu_r(t+z/c)}\right] \\ &= \frac{\mathcal{E}_0}{2}\sum_{r=-(N-1)/2}^{(N-1)/2}\left[e^{-i2\pi(\nu_0+r\Delta\nu_{ax})(t-z/c)} - e^{-i\pi(\nu_0+r\Delta\nu_{ax})(t+z/c)}\right] \\ &= \frac{\mathcal{E}_0}{2}\left[e^{-i2\pi\nu_0(t-z/c)}\left(\frac{\sin N\pi\Delta\nu_{ax}(t-z/c)}{\sin\pi\Delta\nu_{ax}(t-z/c)}\right)\right. \\ &\qquad \left. -e^{-i2\pi\nu_0(t+z/c)}\left(\frac{\sin N\pi\Delta\nu_{ax}(t+z/c)}{\sin\pi\Delta\nu_{ax}(t+z/c)}\right)\right]\end{aligned}$$

By taking the real part of this equation we can write the real electric field of

[1]This is only one possible way mode phases can be locked and is referred to as fundamental mode lock equation. many other possibilities exist. For example if mode phases satisfy the relation $\varphi_{r+1} - \varphi_r = \varphi_r - \varphi_{r-1} + \pi$ is second harmonic locking etc.

the laser as

$$E(z,t) = \frac{\mathcal{E}_0}{2}\left[\cos 2\pi\nu_0(t - z/c)\left(\frac{\sin N\pi\Delta\nu_{ax}(t - z/c)}{\sin \pi\Delta\nu_{ax}(t - z/c)}\right)\right.$$
$$\left. - \cos 2\pi\nu_0(t + z/c)\left(\frac{\sin N\pi\Delta\nu_{ax}(t + z/c)}{\sin \pi\Delta\nu_{ax}(t + z/c)}\right)\right]$$

The first term inside square brackets represents an optical wave of frequency ν_0 traveling to the right with its amplitude modulated by the factor inside parentheses. Similarly, the second term represents a wave with optical frequency ν_0 with modulated amplitude traveling to the left. Wave intensity averaged over an optical cycle is

$$I(z,t) = I_0\left[\left(\frac{\sin N\pi\Delta\nu_{ax}(t - z/c)}{\sin \pi\Delta\nu_{ax}(t - z/c)}\right)^2 + \left(\frac{\sin N\pi\Delta\nu_{ax}(t + z/c)}{\sin \pi\Delta\nu_{ax}(t + z/c)}\right)^2\right] \quad (10.3)$$

Because of modulation, the wave has sharp intensity peaks when

$$\pi\Delta\nu_{ax}(t \mp z/c) = s\pi \quad \Rightarrow \quad \Delta\nu_{ax}(t \mp z/c) = s \quad (10.4)$$

where s is an integer. The peak intensity is $N^2 I_0$. On either side of a peak the intensity falls to zero when

$$\pi\Delta\nu_{ax}(t \mp z/c) = s\pi \pm \pi/N \quad \Rightarrow \quad \Delta\nu_{ax}(t \mp z/c) = s \pm 1/N \quad (10.5)$$

There are secondary intensity peaks (numbering $N - 2$) between two successive intensity maxima given by Eq.(10.4), but the secondary peak intensities are smaller at least by a factor $1/N^2$ relative to the principal peak intensity. Secondary peak intensities are suppressed even smaller when we consider N modes with decreasing amplitudes as we move away from line center. In the rest of this chapter we will ignore these much weaker secondary peaks.

Intensity pulses may be taken to last for a duration defined by the interval between interval

$$\Delta t_p = 2/N\Delta\nu_{ax} = 2\tau_R/N$$

A good approximation to pulse width (FWHM) is

$$\tau_p = \frac{1}{\Delta\nu_{osc}} = \frac{1}{N\Delta\nu_{ax}} = \frac{\tau_R}{N} \quad (10.6)$$

We can then write pulse width to be

$$\tau_p = \frac{1}{\Delta\nu_{osc}}\,. \quad (10.7)$$

This shows that pulses become narrower as the oscillation bandwidth (and, therefore, the number of oscillating modes N) increases.

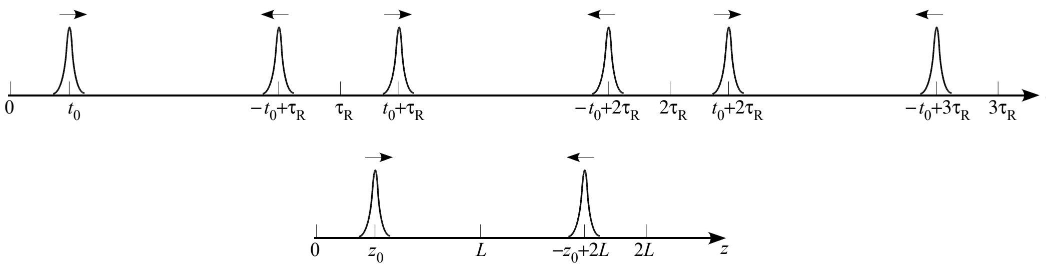

FIGURE 10.2
Pulse sequence observed at a fixed point z_o inside the cavity and pulse location at a fixed instant t_o.

From Eq.(10.4) we find that for given position inside the cavity z_0, we will see right going pulses at times ($t_0 = z_0/c$)

$$t_R = t_0,\ t_0 + \tau_R,\ t_0 + 2\tau_R, \cdots, \tag{10.8}$$

and left going pulses at times

$$t_L = -t_0,\ -t_0 + \tau_R,\ -t_0 + 2\tau_R, \cdots, \tag{10.9}$$

Thus we see a right (or left) going pulse every round trip time τ_R of peak intensity $N^2 I_0$ and duration τ_R/N.

How many pulses are circulating inside the cavity at any time? To answer this question let us look at pulse position a fixed time t_0. Then the right and left going pulse locations are given by ($z_0 = ct_0$)

$$z_R = z_0 \qquad z_L = -z_0 + 2L \tag{10.10}$$

It is easily checked that only one pulse (either right or left going) is present in the cavity ($0 < z < L$) at a given time. Combining the two pictures we see that inside the laser cavity a bouncing bullet picture for a mode-locked laser pulse emerges.

Outside the cavity, for example to the right of mirror M_2, we see an output only when a right traveling wave is incident on M_2. The output field

$$\mathcal{E}_{out} = \sqrt{\frac{\ln \mathcal{R}_2}{(A+T)}}\, \mathcal{E}_0 \left[\frac{\sin N\pi\Delta\nu_{ax}(t - z/c)}{\sin \pi\Delta\nu_{ax}(t - z/c)}\right]^2 \cos[2\pi\nu_0(t - z/c)] \tag{10.11}$$

thus consists of a sequence of short pulses of period τ_R, duration $1/\Delta\nu_{osc}$, and peak intensity $N^2 I_0 \ln \mathcal{R}_2/(A+T)$.

The expression for pulse width that we have derived is valid for an inhomogeneously broadened laser which permits multimode oscillation. The mechanism for producing multimode oscillation in a homogeneously broadened system is different and is discussed in the next section.

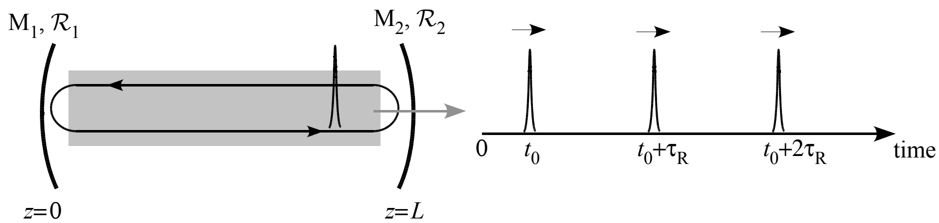

FIGURE 10.3
Output pulse sequence from a mode-locked laser.

In an inhomogeneously broadened laser, mode amplitudes follow a gaussian distribution

$$\mathcal{E}_m^2 = \mathcal{E}_0^2 e^{\left[-(r\Delta\nu_{ax}/\Delta\nu_{osc})^2 4\ln 2\right]}$$

With this mode amplitude distribution, the sum over oscillating modes for the right going wave can be evaluated as

$$\begin{aligned}\mathcal{E}(z,t) &= e^{-i2\pi\nu_0(t-z/c)}\sum_r \mathcal{E}_r e^{-i2\pi r\Delta\nu_{ax}(t-z/c)} \\ &\approx \mathcal{E}_0 e^{-i2\pi\nu_0(t-z/c)}\int_{-\infty}^{\infty} dr e^{\left[-(r\Delta\nu_{ax}/\Delta\nu_{osc})^2 4\ln 2 - i2\pi r\Delta\nu_{ax}(t-z/c)\right]} \\ &= \mathcal{E}_0 e^{-i2\pi\nu_0(t-z/c)}\frac{\Delta\nu_{osc}}{\Delta\nu_{ax}}\sqrt{\frac{\pi}{2\ln 2}}\, e^{-(t-z/c)^2\left(\pi\Delta\nu_{osc}/\sqrt{2\ln 2}\right)^2} \end{aligned} \tag{10.12}$$

Note that this approximation of replacing a discrete sum by an integral is good only for the duration of the pulse. While Eq.(10.12) is a good approximation to pulse shape it washes out the periodic character of mode-locked pulse train predicted by Eq.(10.3) and observed in the experiment. From Eq.(10.12), the pulse intensity is found to be

$$I(z,t) = I_0 e^{-2(t-z/c)^2\left(\pi\Delta\nu_{osc}/\sqrt{2\ln 2}\right)^2}. \tag{10.13}$$

Full width at half maximum of this pulse is given by

$$\tau_p = \frac{2\ln 2}{\pi\Delta\nu_{osc}} = \frac{0.44}{\Delta\nu_{osc}} = \frac{0.44\tau_R}{N} \tag{10.14}$$

The spatial extent of this pulse is $\frac{0.88L}{N}$. Such a pulse is said to be transform limited.

An homogneously broadened laser left to its own devices will oscillate in a single mode. Spatial hole burning and spatial distribution of gain may, however, permit a small number of modes concentrated near the center of the gain profile to oscillate. Thus $\Delta\nu_{osc}$ for an homogneously broadened gain medium is a narrow region near the center of the gain profile. This tendency of a homogneously broadened gain medium to narrow down the oscillation spectrum is opposed by the modulator which tends to broaden the oscillation spectrum by producing sidebands and pushing the oscillation spectrum outward from line center in frequency. Mode locking behavior in an homogeneously broadened laser is therefore a balance between these two opposing tendencies. This is reflected in the expression for mode-locked pulse width in a homogeneous braodened system

$$\tau_p \approx \frac{0.45}{\sqrt{\nu_m \Delta\nu_{osc}}}\,, \tag{10.15}$$

where ν_m is the frequency of modulation and $\Delta\nu_{osc}$ is the oscillation bandwidth.