Laser Physics: PHYS 5734/4734

Spring 2009, Homework Set - 1

Due: Thursday, Feb. 10, 2009.

1(a) Compare the spectral irradiance ($\mathrm{W/m^2 \cdot Hz}$) of a 1 mW He:Ne laser beam (wavelength 633 nm) of diameter 1 mm and linewidth $\delta\nu = 100$ MHz and a black body radiator at a temperature of 6000 K. Recall that the intensity of radiation emitted by a blackbody in a frequency band $\delta\nu$ around a frequency ν is given by

$$I_\nu = \frac{1}{4} c\rho(\nu)\delta\nu = \frac{(2\pi h\nu^2/c^2)\delta\nu}{e^{(h\nu/k_B T)} - 1}$$

Take $T = 6000$ K, $\delta\nu = 100$ MHz and ν corresponding to 633 nm. Note that by writing

$$\rho(\nu) = \left[\frac{8\pi\nu^2}{c^3}\right] \frac{h\nu}{e^{(h\nu/k_B T)} - 1}$$

we can see that $\rho(\nu)$ is the energy density per unit volume per unit frequency interval $[J/m^3 \cdot Hz]$. The quantity inside square brackets is the number of modes per unit volume in a frequency band $\delta\nu$ around the frequency ν and the second factor is the energy of a mode of frequency ν.

1(b) Laser pulses of 0.5 to 25 ns duration carrying 10.4 kJ of UV light have been produced. In these pulses, the peak power can exceed $10^{13}\,W$. Assuming a beam diameter of 1.0 cm estimate the irradiance of these pulses assuming that the intensity is uniform across the beam spot. Given that the irradiance and the electric field amplitude are related by $\frac{1}{2}\epsilon_0\mathcal{E}_0^2 c$, where $\mathcal{E}_0$ is the electric field amplitude, ϵ_0 is the permittivity of free space and c is the speed of light in vacuum. Compare this field strength with the field experienced by an electron in a Hydrogen atom.

2 **Lens Focal Lengths:** The $ABCD$ matrix of a lens (not necessarily thin) is given by

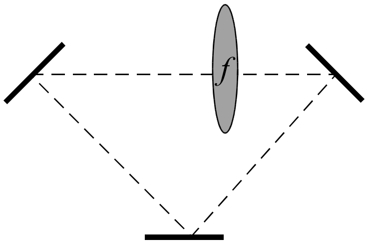
$$\begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix}.$$

Consider a bundle of rays incident parallel to the axis of the lens. Find the location of the image focus (the point where the incident bundle of parallel rays intersects) relative to the exit surface of the lens in terms of the elements of the ABCD matrix of the lens. Do the same for the object focus (the point on the axis of the lens such that rays diverging it are rendered parallel after passing through the lens). Check your answers using the ABCD matrix for a thin lens derived in class. How will things be different for a thick lens?

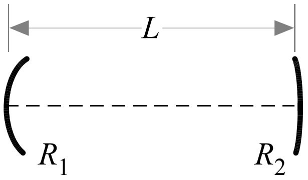
3(a) **Thick Lens:** For a thick lens with radii of curvature R_1 and R_2 for the entrance and exit surfaces, respectively, refractive index n and thickness d, obtain the ABCD matrix assuming the medium surrounding the lens to be air.

(b) Find the location of its object and image focal points if $n = 1.58$, $R_1 = 10$ cm, $R_2 = -20$ cm, and $d = 2.0$ cm. Include a sketch to illustrate your answer.

4 **Ring Cavity:** A triangular ring cavity of perimeter L is formed by three plane mirrors with a positive lens (focal length f) inserted between two of the mirrors. Find a relation bewteen L and f for the cavity to be stable.



5 **Symmetric Two Mirror cavity:** Consider a two-mirror cavity formed by two identical concave mirrors ($R_1 = R_2 = R$) facing each other and separated by a distance L. Find the range of value of L for which the cavity is stable.



6 **Two Mirror Cavity Stability:** A two mirror resonator is formed by a **convex** mirror of radius $R_1 = 1.0$ m and a **concave** mirror of radius $R_2 = 1.5$ m. What is the maximum and minimum possible mirror separation if this is to remain a stable resonator?

1 Solutions

1. Compare the spectral irradiance (W/m^2·Hz) of a 1 mW He:Ne laser beam (wavelength 633 nm) of diameter 1 mm and linewidth $\delta\nu = 100$ MHz and a black body radiator at a temperature of 6000 K. Recall that the intensity of radiation emitted by a blackbody in a frequency band $\delta\nu$ around a frequency ν is given by

$$I_\nu = \frac{1}{4}c\rho(\nu)\delta\nu = \frac{(2\pi h\nu^3/c^2)\delta\nu}{e^{(h\nu/k_BT)} - 1}$$

Take $T = 6000$ K, $\delta\nu = 100$ MHz and ν corresponding to 633 nm. Note that by writing

$$\rho(\nu) = \left[\frac{8\pi\nu^2\delta\nu}{c^3}\right]\frac{h\nu}{e^{(h\nu/k_BT)} - 1}$$

we can see that $\rho(\nu)$ is the energy density per unit volume per unit frequency interval $[J/m^3 \cdot Hz]$. The quantity inside square brackets is the number of modes per unit volume in a frequency band $\delta\nu$ around the frequency ν and the second factor is the energy of a mode of frequency ν.

Solution: Assuming uniform field distribution over the diameter of the beam, the intensity will be

$$\begin{aligned} I_{\text{He:Ne}} &= \frac{\text{power}}{\text{area}} = \frac{10^{-3}\,\text{W}}{\pi(0.5\times10^{-3})^2\text{m}^2} = 1.27\times10^3\,\text{W/m}^2 \\ &\approx 1.3\times10^2\,\text{mW/cm}^2 \end{aligned}$$

We compare this to the intensity from a blackbody at $T = 6000$ K in a $\delta\nu = 100$ MHz frequency band around the laser frequency (or the wavelength $\lambda = 0.633\,\mu$m)

$$I_{\text{BB}} = \frac{(2\pi h\nu^3/c^2)\delta\nu}{e^{(h\nu/k_BT)} - 1} = \frac{2\pi hc}{\lambda^3}\frac{1}{e^{(h\nu/k_BT)} - 1}\delta\nu\,.$$

Using the values (MKS units)

$$\begin{aligned} \frac{h\nu}{k_BT} &= \frac{1}{k_BT}\frac{hc}{\lambda} = \frac{1}{1.38\times10^{-23}\times6000}\cdot\frac{6.626\times10^{-34}\times3.0\times10^8}{0.633\times10^{-6}} \\ &= 3.79 \approx 3.8 \\ e^{(h\nu/k_BT)} - 1 &= 43 \end{aligned}$$

we find the intensity from a black body

$$\begin{aligned} I_{\text{BB}} &= \frac{2\pi\times(6.626\times10^{-34})\times(3.0\times10^8)}{(0.633\times10^{-6})^3}\cdot\frac{1}{43}\cdot10^8\,,\text{W/m}^2 \\ &= 11.45\,\text{W/m}^2 \approx 1.1\,\text{mW/cm}^2 \end{aligned}$$

Thus the intensity of radiation from a blackbody at the surface temperature of the sun is 100 times smaller than the intensity of a low power He:Ne laser in the emission bandwidth of the laser.

Note 1: Intensity is not a good quantity to compare because it can be changed by focusing or defocusing the beam. Further, the bandwidth of a laser may vary. For example the emission bandwidth of a He:Ne laser can vary from about 1000 MHz to 0.1 Hz!

Note 2: In estimating various quantities you may find the following combinations of fundamental constants easier to remember rather than the constants themselves:

$$\begin{aligned}
1\text{eV} &= 1.602 \times 10^{-19}\,\text{J} \approx 1.6 \times 10^{-19}\,\text{J} \\
\hbar c &= 197\,\text{MeV-fm} \approx 200\,\text{eV-nm} \qquad 1\,\text{nm} = 10^{-9}\,\text{m} \\
hc &= 1238\,\text{eV-nm} \approx 1.25\,\text{eV-}\mu\text{m} \qquad 1\,\mu\text{m} = 10^{-6}\,\text{m} \\
k_{\text{B}} \times (300K) &= 2.58 \times 10^{-2}\,\text{eV} \approx \frac{1}{40}\,\text{eV} \qquad 300\,\text{K is room temperature (27 C)}
\end{aligned}$$

You will get your answers within a few percent of the "exact" values. As an example consider the estimate of I_{BB}

$$\begin{aligned}
\frac{h\nu}{k_B T} &\approx \frac{hc}{\lambda}\frac{1}{k_B \times 300(T/300)} = \frac{1.25}{0.633} \times \frac{40}{20} = 3.9 \\
e^{(h\nu/k_B T)} - 1 &\approx 49
\end{aligned}$$

This leads to

$$\begin{aligned}
I_B &= \frac{2\pi \times 1.25\,\text{eV-}\mu\text{m}}{(0.633\,\mu\text{m})^3} \cdot \frac{1}{49} \cdot 10^8 = 0.632 \times 10^8\,\text{eV}/\mu\text{m}^2\text{-s} \\
&= (0.632 \times 10^8) \times 1.6 \times 10^{-19} \times 10^{12}\,\text{W/m}^2 \\
&= 10.1\,\text{W/m}^2 \approx 1.0\,\text{mW/cm}^2
\end{aligned}$$

This is approximately 100 times smaller than the intensity of the laser.

1(b) Laser pulses of 0.5 to 25 ns duration carrying 10.4 kJ of UV light have been produced. In these pulses, the peak power can exceed $10^{13}\,W$. Assuming a beam diameter of 1.0 cm estimate the irradiance of these pulses assuming that the intensity is uniform across the beam spot. Given that the irradiance and the electric field amplitude are related by $\frac{1}{2}\epsilon_0\mathcal{E}_0^2 c$, where $\mathcal{E}_0$ is the electric field amplitude, ϵ_0 is the permittivity of free space and c is the speed of light in vacuum. Compare this field strength with the field experienced by an electron in a Hydrogen atom.

Solution: The irradiance is estimated to be

$$I = \frac{P}{\pi d^2/4} = \frac{10^{13} \times 4}{\pi \times (10^{-2})^2} = 1.27 \times 10^{17}\,\text{W/m}^2 \approx 1.3 \times 10^{13}\,\text{W/cm}^2$$

Equating this to the expression for the irradiance in terms of the electric field we find

$$\begin{aligned}
I = \frac{1}{2}\epsilon_0\mathcal{E}_0^2 c \quad &\Rightarrow \quad \mathcal{E}_0 = \sqrt{\frac{2I}{\epsilon_0 c}} \\
\text{or} \quad &\mathcal{E}_0 = \sqrt{\frac{8\pi I}{4\pi\epsilon_0 c}} \\
&= \sqrt{\frac{8\pi \times 9.0 \times 10^9 \times 1.27 \times 10^{17}}{3.0 \times 10^8}} \\
&= 9.78 \times 10^9\,\text{V/m} \approx 9.8 \times 10^7\,\text{V/cm}
\end{aligned}$$

In a hydrogen atom, the electric field experienced by an electron in the ground state will be

$$\mathcal{E}_{at} = \frac{e}{4\pi\epsilon_0 a_0^2}$$

where $a \approx 0.53 \times 10^{-10}$ m is Bohr radius [See, for example, Milonni and Eberly] and $e = 1.60 \times 10^{-19}$ C is the charge (magnitude) of a proton. Using this value we find

$$\mathcal{E}_{at} = \frac{1.6 \times 10^{-19} \times 9.0 \times 10^{9}}{(0.53 \times 10^{-10})^2} = 5.1 \times 10^{11}\,\mathrm{V/m} = 5.1 \times 10^{9}\,\mathrm{V/cm}$$

This fields is about 50 times the laser field. We shall see later that by focusing a high power laser beam, elctric fields millions of times stronger than the field experienced by an atomic electron can be produced.

2(b) Lasers with frequency stability better than $\Delta\nu = 10^{-2}$ Hz have been designed. Estimate the corresponding wavelength stability, coherence time and coherence length for the light generated by such a laser. Compare these numbers with those for the light emitted by a sodium lamp with $\Delta\nu = 9 \times 10^{9}$ Hz.

Using the bandwidth and coherence time relation $2\pi\Delta\nu\tau_c \approx 1$ and the coherence length expression $\ell_c = c\tau_c$, we obtain

Laser	Sodium Lamp
$\tau_c = \frac{1}{2\pi\Delta\nu} = 15.9 \approx 16\,\mathrm{s}$	$\tau_c = \frac{1}{2\pi\Delta\nu} = 1.76 \times 10^{-11} \approx 18\,\mathrm{ps}$
$\ell_c = c\tau_c = 4.8 \times 10^{9}\,\mathrm{m}$	$\ell_c = 5.3 \times 10^{-3}\,\mathrm{m} = 5.3\,\mathrm{mm}$
Exceeds earth-moon distance!	

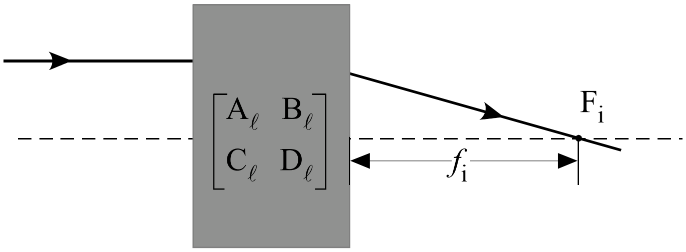
2 **Lens Focal Lengths:** The $ABCD$ matrix of a lens (not necessarily thin) is given by

$$\begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} .$$

Consider a bundle of rays incident parallel to the axis of the lens. Find the location of the image focus (the point where the incident bundle of parallel rays intersects) relative to the exit surface of the lens in terms of the elements of the ABCD matrix of the lens. Do the same for the object focus (the point on the axis of the lens such that rays diverging it are rendered parallel after passing through the lens). Check your answers using the ABCD matrix for a thin lens derived in class. How will things be different for a thick lens?

(a) **Solution:**A lens has two focal points.

Image focal point F_i is the point where an arbitrary ray incident parallel to the axis of the lens intersects the axis after emerging from the lens. Let f_i (to be determined) be the distance of F_i measured from the exit surface of the lens. To reach the image focal point F_i the ray must travel an extra distance f_i in air after emerging from the lens [Figure]. Upon noting that the slope of a ray incident parallel to the axis is $r_i' = 0$ and the displacement of a ray where it intersects the axis of the lens is $r_o = 0$, it is clear that the incident and outgoing rays are



$$\tilde{r}_i = \begin{bmatrix} r_i \\ 0 \end{bmatrix} \qquad \tilde{r}_o = \begin{bmatrix} 0 \\ r_o' \end{bmatrix}$$

where r_i is arbitrary. These rays are related by

$$\begin{bmatrix} 0 \\ r_o' \end{bmatrix} = \begin{bmatrix} 1 & f_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} \begin{bmatrix} r_i \\ 0 \end{bmatrix} = \begin{bmatrix} [A_\ell + f_i C_\ell] r_i \\ C_\ell r_i \end{bmatrix} .$$

Equating the first elements on the two sides of this equation we find, since r_i is arbitrary,

$$f_i = -\frac{A_\ell}{C_\ell}$$

$$\text{or} \quad \frac{1}{f_i} = -\frac{C_\ell}{A_\ell} .$$

Thus the image focal point F_i is located a distance f_i to the right of the exit surface as shown in the Figure. Where is F_i located if f_i comes out to be negative?

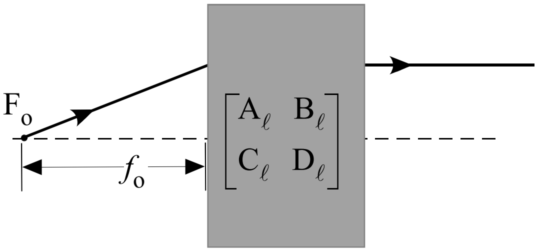


Figure 2

Object focal point F_o [Fig. (2)] is the point on the axis of a lens such that a ray starting from this point (with arbitrary slope in the paraxial approximation) and incident on the lens emerges to the axis of the lens. Let f_o (to be determined) be the distance of F_o from the entrance face of the lens. To reach the lens the ray travels a distance f_o in air. Upon noting that the slope of a ray parallel to the axis is 0 and the displacement of a ray where it intersects the axis of the lens is 0, it is clear that the incident and outgoing rays are

$$\tilde{r}_i = \begin{bmatrix} 0 \\ r_i' \end{bmatrix} \qquad \tilde{r}_o = \begin{bmatrix} r_o \\ 0 \end{bmatrix}$$

are related by

$$\begin{bmatrix} r_o \\ 0 \end{bmatrix} = \begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} \begin{bmatrix} 1 & f_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_i' \end{bmatrix} = \begin{bmatrix} (f_oA_\ell + B_\ell)r_i' \\ (f_oC_\ell + D_\ell)r'{}_i \end{bmatrix}$$

Equating the second elements from the two sides, we find

$$f_o = -\frac{D_\ell}{C_\ell} \quad \text{or} \quad \frac{1}{f_i} = -fracC_\ell D_\ell$$

Thus the object focal point F_o of the lens is located a distance f_o given by this expression to the left of the entrance surface of the lens in Figure 2.

For a thin lens we can neglect lens thickness d. We then obtain (as shown in class)

$$\begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{(1-n)}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{(n-1)}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 \end{bmatrix}.$$

Using the expressions for $A_\ell = 1 = D_\ell$ and C_ℓ we find the well known result

$$\frac{1}{f_i} = -C_\ell = \frac{1}{f_o} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Since $f_o = f_i$ we can use them the same symbol f and speak of it as simply the focal length of the lens.

Note: For a thick lens ($d \neq 0$), the distances of the two focal points from the respective lens surfaces are different. If, however, we measure the object focal point distance from the **primary principal plane** and the image focal point distance from the **secondary principal plane**) [see, for example, *Optics*, E. Hecht (Addison Wesley, Reading, Mass., 1987), Chapter 6] of the lens we find that the two focal lengths are equal and given by $f = -1/C_\ell$. For a lens in air the primary principal plane is located at a distance $h_1 = \frac{A_\ell - 1}{C_\ell}$ from the first lens surface and the secondary principal plane is located at a distance $h_2 = \frac{D_\ell - 1}{C_\ell}$ from the second lens surface.

3(b) **Thick Lens:** For a thick lens with radii of curvature R_1 and R_2 on the entrance and exit surfaces, an refractive index n, and a thickness d, (a) obtain the ABCD matrix for the lens. (b) Find the location of its [both object and image] focal points if $n = 1.58$, $R_1 = 10$ cm, $R_2 = -20$ cm, and $d = 2.0$ cm. (c) Draw a sketch to illustrate your answer.

(b) **Solution: Lens Matrix**

In passing the lens a ray encounters three elements [Figure 1]

1. air-glass interface (radius of curvature R_1)
2. straight section of thickness d in glass
3. glass-air interface (radius of curvature R_2) The ABCD matrix of the lens is then $M = M_3 M_2 M_1$, where M_i is the matrix of element i. Using the results derived in class we find

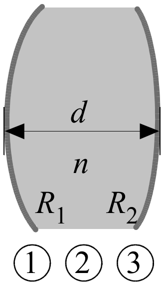


Figure 1

$$\begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{(1-n)}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{(n-1)}{nR_1} & \frac{1}{n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{d(n-1)}{nR_1} & \frac{d}{n} \\ -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right] & 1 + \frac{d(n-1)}{nR_2} \end{bmatrix}$$

(b) For a lens with given parameters we have

$$\begin{bmatrix} A_\ell & B_\ell \\ C_\ell & D_\ell \end{bmatrix} = \begin{bmatrix} 1 - \frac{d(n-1)}{nR_1} & \frac{d}{n} \\ -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right] & 1 + \frac{d(n-1)}{nR_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9265 & 1.2658\,\mathrm{cm} \\ -0.08487\,\mathrm{cm}^{-1} & 0.9633 \end{bmatrix}$$

Using the formulas of the preceding problem, we then find

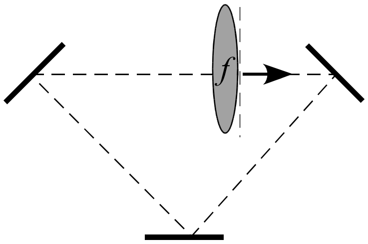
$$f_i = -\frac{A_\ell}{C_\ell} = -\frac{0.9265}{-0.08487}\,\mathrm{cm} = 10.9\,\mathrm{cm}\,,$$

$$f_o = -\frac{D_\ell}{C_\ell} = -\frac{0.9633}{-0.08487}\,\mathrm{cm} = 11.4\,\mathrm{cm}\,.$$

Thus the image focal point F_i lies 10.9 cm to the right of the exit surface and the object focal point F_o lies 11.4 cm to the left of the entrance surface in Fig. 1.

Note: The distances of the two focal points from the respective lens surfaces are different. If we measure the object focal point distance from the **primary principal plane** and the image focal point distance from the **secondary principal plane**) [see, for example, *Optics*, E. Hecht (Addison Wesley, Reading, Mass., 1987), Chapter 6] of the lens we find that the two focal lengths are equal and given by $f = -1/C_\ell = 11.8$ cm. For a lens in air the primary principal plane is located at a distance $h_1 = \frac{A_\ell - 1}{C_\ell}$ from the first lens surface and the secondary principal plane is located at a distance $h_2 = \frac{D_\ell - 1}{C_\ell}$ from the second lens surface.

5(a) **Ring Cavity:** A triangular ring cavity of perimeter L is formed by three plane mirrors with a positive lens (focal length f) inserted between two of the mirrors. Find a relation bewteen L and f for the cavity to be stable.



Solution : The ring is equivalent to a periodic system with the basic block consisting of one round trip around the resonator. Then starting just to the right of the lens and proceeding clockwise, the round trip matrix is

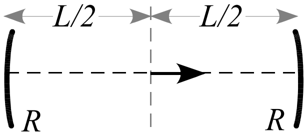
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & L \\ -\dfrac{1}{f} & 1 - \dfrac{L}{f} \end{bmatrix} \tag{1}$$

Here we have used the fact that the ABCD matrix for a plane mirror is a unit matrix. Using the stability condition $-1 \leq \frac{1}{2}(A+D) \leq 1$, we find

$$\begin{aligned} -1 \leq \frac{1}{2}(A+D) \leq 1 \quad \Rightarrow & \quad -1 \leq 1 - \frac{L}{2f} \leq 1 \\ \text{or} & \quad 0 \leq 2 - \frac{L}{2f} \leq 2 \\ \text{or} & \quad 0 \leq 1 - \frac{L}{4f} \leq 1 \\ \text{or} & \quad 0 \leq L \leq 4f \end{aligned}$$

Thus the focal length of the lens must be positive and the ring perimeter must be less than $4f$.

5(b) **Symmetric Two Mirror cavity :**Consider a two-mirror cavity formed by two identical mirrors ($R_1 = R_2 = R$) facing each other and separated by a distance L. For what values of L is the cavity stable?



Solution: This resonator is equivalent to a periodic system with the basic block consisting of a trip around the resonator. Starting to the right at the reference plane (in the middle of this resonator) shown in the figure, the round trip ABCD matrix is

$$\begin{aligned}
\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & \end{bmatrix} \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L \\ -\frac{2}{R} & 1 - \frac{2L}{R} \end{bmatrix} \begin{bmatrix} 1 & \frac{L}{2} \\ -\frac{2}{R} & 1 - \frac{L}{R} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - \frac{2L}{R} & \frac{3}{2}L - \frac{L^2}{R} \\ -\frac{4}{R} + \frac{4L}{R^2} & 1 - \frac{4L}{R} + \frac{2L^2}{R^2} \end{bmatrix} \\
&= \begin{bmatrix} 1 - \frac{4L}{R} + \frac{2L^2}{L^2} & 2L - \frac{2L^2}{R} + \frac{L^3}{R^2} \\ -\frac{4}{R} + \frac{4L}{R^2} & 1 - \frac{4L}{R} + \frac{2L^2}{R^2} \end{bmatrix}
\end{aligned}$$

The stability condition $-1 \leq \frac{1}{2}(A + D) \leq 1$ for this matrix gives

$$-1 \leq 1 - \frac{4L}{R} + \frac{2L^2}{R^2} \leq 1$$

$$\text{or} \qquad 0 \leq 1 - \frac{2L}{R} + \frac{L^2}{R^2} \leq 1 \qquad \text{adding 1 and dividing by 2}$$

$$\text{or} \qquad 0 \leq \left(1 - \frac{L}{R}\right)^2 \leq 1 \qquad \text{completeing the square}$$

We could have written this down directly by using the result derived in class. An inspection of this equation shows that this inequality is satisfied if the factor $1 - L/R$ lies between -1 and 1,

$$\text{or} \qquad -1 \leq \left(1 - \frac{L}{R}\right) \leq 1$$

$$\text{or} \qquad 0 \leq \left(1 - \frac{L}{2R}\right) \leq 1 \qquad \text{adding 1 and dividing by 2}$$

$$\text{or} \qquad 0 \leq L \leq 2R$$

Thus R must be poitive and the mirror separation L must not exceed $2R$. Note that $L = 2R$ is the concentric spherical resonator.

6 **Two Mirror Cavity Stability :** A two mirror resonator is formed by a convex mirror of radius $R_1 = -1.0$ m and a concave mirror of radius $R_2 = 1.5$ m. What is the maximum and minimum possible mirror separation if this is to remain a stable resonator?

Solution: The condition for the stability of a resonator with mirrors of radii of curvature R_1 and R_2 separated by a distance L is

$$0 \le g_1 g_2 \le 1$$

where $g_1 = 1 - \frac{L}{R_1}$ and $g_2 = 1 - \frac{L}{R_2}$. Recall that while R_1 and R_2 can be positive or negative, L is always positive. For the given resonator ($R_1 = -1.0$ m and $R_2 = 1.5$ m) the condition of stability becomes

$$0 \le (1+L)(1-2L/3) \le 1$$

where all lengths are in meters. The lower limit of stability requires

$$\begin{aligned} 0 \le (1+L)(1-2L/3) \quad &\Longrightarrow \quad 0 \le (1-2L/3) \\ &\text{or} \quad L \le \frac{3}{2} \end{aligned} \tag{1}$$

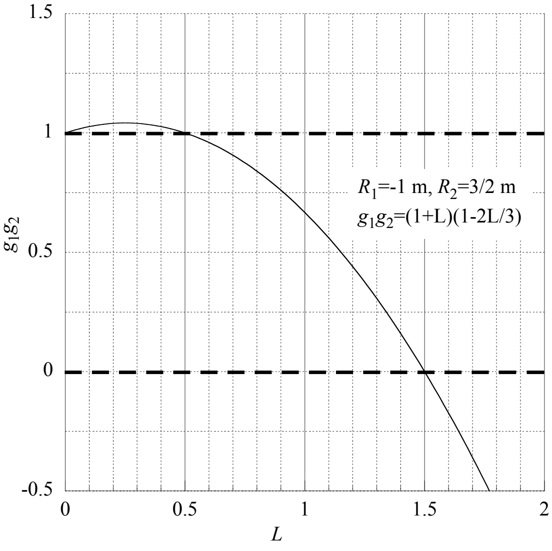
The upper limit of stability requires

$$\begin{aligned} (1+L)(1-2L/3) \le 1 \quad &\Longrightarrow \quad 1 + \frac{L}{3} - \frac{2L^2}{3} \le 1 \\ &\Rightarrow \quad \frac{L}{3}(1-2L) \le 0 \\ &\Rightarrow \quad L \le 0 \quad \text{or} \quad L \ge \frac{1}{2} \end{aligned} \tag{2}$$

Since $L \le 0$ is unphysical, $L \ge \frac{1}{2}$ sets the limit of stability. Combining Eqs. (1) and (2) we find that stability is achieved when L lies in the range

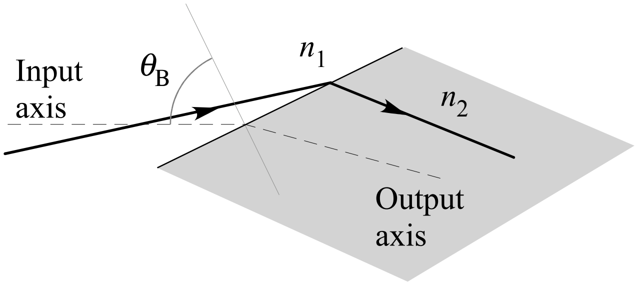
$$\frac{1}{2} \le L \le \frac{3}{2}\,. \tag{3}$$

We can also arrive at this result graphically by plotting $g_1 g_2 = (1+L)(1-2L/3)$ as a function of L. For stability, $g_1 g_2$ must lie between the limits 0 and 1 (dashed lines).



4 **Incidence Near Brewster's Angle :** Certain optical elements (tuning plates, gain rods etc.) require that they be used near Brewster's angle of incidence to minimize reflection losses. In such cases the ray transformation matrices in the plane of incidence (tangential plane) and the plane perpendicular to it (sagittal plane) are different. (See the table of ABCD matrices in the notes). Consider an interface at Brewster's angle between two media of refractive indices n_1 and n_2. The figure below shows the plane of incidence (tangential plane). Find its ABCD matrix in the tangential and sagitttal planes.

Hint: Use special cases of the matrices given in the notes appropriate to your problem.



Solution: For an arbitrary angle of incidence on a plane interface ($R \to \infty$), the ABCD matrices in the two planes reduce to

$$\text{Sagittal plane} \qquad \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{n_1}{n_2} \end{bmatrix}$$

$$\text{Tangential plane} \qquad \begin{bmatrix} \dfrac{\cos\theta_2}{\cos\theta_1} & 0 \\ 0 & \dfrac{n_1\cos\theta_1}{n_2\cos\theta_2} \end{bmatrix}$$

Note that in this case we are dealing with rays that are close to the optical axis (dashed line) which meets the interface at an arbitrary angle θ_1. Ray displacement is measured from this axis and slope is the (small) angle that the ray makes with the axis.

At Brewster's angle $\theta_1 + \theta_2 = \pi/2$ so that $\theta_1 = \pi/2 - \theta_2$ and $\theta_2 = \pi/2 - \theta_1$. Using this in Snell's law, we find

$$\begin{aligned} & n_1 \sin\theta_1 = n_2 \sin\theta_2 \\ \text{or} \quad & n_1 \cos\theta_2 = n_2 \cos\theta_1 \\ \text{or} \quad & \frac{\cos\theta_1}{\cos\theta_2} = \frac{n_1}{n_2} \end{aligned}$$

The matrix for the tangential plane then becomes

$$\begin{bmatrix} \dfrac{n_2}{n_1} & 0 \\ 0 & \dfrac{n_1^2}{n_2^2} \end{bmatrix}$$